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Expanded Lyapunov-Krasovskii Functionals and Stability Analysis in Delayed Neural Networks via Augmented Zero Equality Approach

Seung-Ho Kim^{ID}, Seung-Hoon Lee*^{ID}, Myeong-Jin Park^{ID}, Oh-Min Kwon*^{ID}, and Jun-Min Park^{ID}

Abstract: This paper proposes improved Lyapunov-Krasovskii functionals (LKFs) for asymptotic stability of generalized neural networks (GNNs) with time-varying delays. By utilizing generalized free-weighting matrix inequality (GFWMI) and some mathematical techniques, sufficient conditions which are dependent on the size of time delays are derived for guaranteeing the stability of GNNs. Additionally, the augmented zero equality approach (AZE) is applied to enhance the results and eliminate the free variables. Three numerical examples show that the proposed method can be effective and provide less conservative results than previous researches.

Keywords: Generalized neural networks, Lyapunov-Krasovskii functionals, stability analysis, time-varying delay, zero equality.

1. INTRODUCTION

For recent two decades, neural networks have attracted many attentions for stability analysis with time-varying delays in science and engineering fields such as pattern recognition, secure communication, image processing, associative memories, etc [1]. It is well known that the development of neural networks with time-varying delays influences dynamic systems. Therefore, the study of neural networks is an influential part that can be used in various fields. For detail, many researchers illustrate the importance of neural networks [2,3].

Time-delay frequently occurs due to the finite switching speed of amplifiers and the communication among neurons in neural networks. It can give poor performances or instability such as oscillations or destruction for a system [4]. Therefore, there is a need to solve these problems, and the study on neural networks with time-delay has occupied one of the critical areas. Recently, several researches have been actively strived to derive the enhanced delay-dependent stability conditions for a time-delay system since they can lead to less conservative results than the delay-independent ones [2,5,6].

In the time-delay system, it is usually practical to obtain stability conditions that utilize Lyapunov stability the-

ory. This crucial method uses tractable linear matrix inequalities (LMIs). LMIs technique helps the stability condition to check easily by solving a certain convex optimization problem. To get the improved effect, the following three items are typically required in Lyapunov theory. One is constructing the proper LKFs. Another is devising an integral inequality that provides lower bounds for integral terms. The other is utilizing some mathematical techniques to improve the stability results.

Firstly, it is well known that constructing the proper LKFs is a critical and very important job to enlarge the maximum delay bounds. Significantly, augmented LKFs can derive less conservative results by providing various cross-terms of the augmented vectors. Through the newly proposed augmented LKFs, including time-varying delays, the advantages of enhanced stability results for GNNs have been represented in [7]. Until recently, the augmented LKFs usually have been used in a variety of systems such as neural networks [8,9], T-S fuzzy systems [10], sampled-data control systems [11], and so on.

Secondly, the integral inequality converts the quadratic form to LMIs and more rigorously estimates the derivative of the LKF for yielding less conservative results. Before two decades, Jensen's inequality has utilized a tighter estimating method in the works [12-14]. After that, Wirtinger-

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based inequality (WBI), which shows improved results than Jensen's inequality, was proposed in [15]. This technique has been widely used in various fields, and the stability analysis for neural networks with time-varying delays (NNTVDs) has also been used [9,16,17]. After the WBI, many recent researches, such as generalized integral inequality, auxiliary function-based inequality, free-matrix-based inequality, and so on, have been introduced in [18-22]. Very recently, the GFWMI has been a main key to expand less conservative regions for NNTVDs [8,23,39].

Finally, the following mathematical techniques derive more influential stability conditions. A zero equality [24] has been extensively utilized in many works [13,25,26]. The reciprocally convex approach is another mathematical technique that gets noticeable attention [27]. This technique can get a lower bound of integral terms and be critical to derive less conservative conditions. Since then, several improved methods have been introduced in NNTVDs [25,28,29,38]. The generalized zero equality approach [30] utilizes the relationship between the integral terms and their integration by parts. Using zero equality can lead to the improved results [8,31]. But, this technique has a disadvantage: it has many free variables that are proportional to the size of the augmented vector. Recently, the AZEA [32] was proposed to eliminate free variables of the zero equalities and to obtain less conservative results. Previous studies on NNTVDs have many free variables since they used zero equalities created using the relationship of augmented vectors. By utilizing the AZEA, the number of free variables is reduced and less conservative results are obtained.

Motivating the above elements, the effectiveness of proper LKFs is intended to verify. Previous studies have not included the activation function terms in LKFs well. However, the LKFs containing the activation function terms are helpful to expand the maximum delay bounds. Significantly, the combined use of some mathematical techniques, the integral inequalities such as GFWMI and the recently proposed AZEA, makes interesting results like maximizing the feasible region in NNTVDs.

The main contributions of this paper can be summarized as below:

- 1) In many previous studies, activation function-based integral terms such as $\int_{t-\varphi_M}^t f(W_2 \varkappa(s))ds$ and $\int_{t-\varphi_M}^t \int_s^t f(W_2 \varkappa(u))duds$ were not included in the augmented vector. With this point, Theorem 1 proposes a differentiated method from other papers. By constructing the augmented LKFs, a wider feasible region can be obtained. In particular, the proposed technique leads to a special utilization in the integral inequality, resulting in more improved results.
- 2) Positiveness derives less conservative stability conditions by utilizing the concept of Lyapunov stabil-

ity. The effect of positiveness utilized in the proposed LKFs is shown in Theorem 2. Some mathematical techniques, such as zero equalities and AZEA, induce a great effect. These methods are critical techniques not only to eliminate the free variables with zero equalities and but also less conservative results in Theorem 3.

With the new LKFs considering the activation function-based terms and GFWMI, stability conditions of a delay-dependent are derived for GNNs in Theorem 1. By including the positiveness of the proposed LKFs, it can be deduced the improved results of feasible region in Theorem 2. Furthermore, by utilizing some mathematical techniques such as some zero equalities and AZEA, the following notions are proposed in Theorem 3. By comparing the results of Theorem 3 and Corollary 1, the proposed LKFs are proved their superiority. Finally, three numerical examples verify the validity and distinction of the proposed results in Theorems 1-3.

Notation: Let us define the used notation in this paper. $\mathbb{R}^n$ and $\mathbb{R}^{n \times l}$ mean the Euclidean space and a real matrix with $n \times l$ dimension, respectively. $\mathbb{S}^n(\mathbb{D}^n)$ and $\mathbb{S}_+^n(\mathbb{D}_+^n)$ represent a symmetric(diagonal) and a symmetric(diagonal) positive definite with $n \times n$ dimension, respectively. I_n means the identity matrix with $n \times n$ dimension. 0_n and $0_{n \times l}$ mean the zero matrix with $n \times n$ and $n \times l$ dimensions. $\star$ means the elements below the main diagonal of a symmetric matrix. $X_\perp$ denotes a basis for the null space of X . $\text{diag}\{d_1, \dots, d_n\}$ stands for the block diagonal matrix with $n \times n$ dimension. $\text{Sym}\{X\}$ denotes that $X + X^T$. $M_{[f(t)]}$ denotes that $M_{[f(t)]} = M_1 + f(t)M_2$ since it is affinely dependent for $f(t)$.

2. PROBLEM STATEMENTS

Let us consider the following system for NNTVDs

$$\begin{aligned} \dot{\varkappa}(t) = & -A\varkappa(t) + W_0 f(W_2 \varkappa(t)) \\ & + W_1 f(W_2 \varkappa(t - \varphi(t))), \end{aligned} \quad (1)$$

$\forall t > 0$, where $\varkappa(t) \in \mathbb{R}^n$ is the neuron state vector, $f(\cdot) \in \mathbb{R}^n = [f_1(\cdot), f_2(\cdot), \dots, f_n(\cdot)]^T$ is the activation function of the neurons, $A \in \mathbb{R}^{n \times n}$ is a positive diagonal matrix, W_0, W_1 , and $W_2 \in \mathbb{R}^{n \times n}$ are the interconnection matrices between neurons. $\varphi(t)$ is a continuous function which means the time-varying delays satisfying

$$0 \leq \varphi(t) \leq \varphi_M, \quad \dot{\varphi}(t) \leq \varphi_D, \quad (2)$$

where φ_M is a positive constant and φ_D is any scalar.

The activation function $f(\cdot)$ is satisfied with the following assumption.

Assumption 1: The activation functions $f_i(\cdot)$ ($i = 1, \dots, n$) with $f_i(0) = 0$ are considered continuous and assumed

to be bounded. They are satisfied with the following condition:

$$\gamma_i^- \leq \frac{f_i(b) - f_i(a)}{b - a} \leq \gamma_i^+, \quad a \neq b \in \mathbb{R}, \quad (3)$$

where γ_i^- and γ_i^+ are real constants.

Remark 1: The concept of Assumption 1 was explained from [33]. Activation function $f(\cdot)$ contains many activation conditions.

This paper aims to derive the delay-dependent stability analysis of the system (1) with time-varying delays. By constructing LKFs and utilizing some mathematical techniques, enhanced stability conditions will be derived in the forms of LMIs.

The following lemmas will be utilized to obtain the main results:

Lemma 1 [34]: For a continuous function $\{\mathcal{X}(s) | s \in [a, b]\} \rightarrow \mathbb{R}^n$, if a positive definite matrix N satisfies, the following integral inequality holds:

$$\begin{aligned} & (b-a) \int_a^b \mathcal{X}^T(s) N \mathcal{X}(s) ds \\ & \geq \left(\int_a^b \mathcal{X}(s) ds \right)^T N \left(\int_a^b \mathcal{X}(s) ds \right). \end{aligned}$$

Lemma 2 [35]: For a continuous function $\{\mathcal{X}(s) | s \in [a, b]\} \rightarrow \mathbb{R}^n$, a given positive definite matrix Q , any matrices S and U with appropriate dimensions and the following integral inequality holds:

$$\begin{aligned} & - \int_a^b \mathcal{X}^T(s) Q \mathcal{X}(s) ds \\ & \leq \text{Sym} \{ \omega_0^T S \omega_1 + \omega_0^T U \omega_2 \} + (b-a) \omega_0^T Q \omega_0, \end{aligned}$$

where $Q = SQ^{-1}S^T + \frac{1}{3}UQ^{-1}U^T$, ω_0 is any vector, $\omega_1 = \int_a^b \mathcal{X}(s) ds$, and $\omega_2 = -\int_a^b \mathcal{X}(s) ds + \frac{2}{b-a} \int_a^b \int_s^b \mathcal{X}(u) du ds$.

Lemma 3 [36]: Let $\varpi \in \mathbb{R}^n$, $\Pi \in \mathbb{S}^n$, and $M \in \mathbb{R}^{m \times n}$. The statements of equivalent follow:

- i) $\varpi^T \Pi \varpi < 0, \forall M \varpi = 0, \varpi \neq 0$,
- ii) $(M^\perp)^T \Pi (M^\perp) < 0$,
- iii) $\exists X \in \mathbb{R}^{n \times m} : \Pi + \text{Sym}\{XM\} < 0$.

Lemma 4 [37]: For each $s \in S$, let $S \subseteq \mathbb{R}^d$, $\Pi : S \rightarrow \mathbb{S}^n$, and $M : S \rightarrow \mathbb{R}^{m \times n}$. The statements of equivalent follow:

- i) $\varpi^T \Pi(s) \varpi < 0, \forall M \varpi = 0, \varpi \neq 0$,
- ii) $(\exists \mu(s) \in \mathbb{R}) : \Pi(s) - \mu(s) M^T(s) M(s) < 0$,
- iii) If there exists $X(s) \in \mathbb{R}^{n \times m}$, $\Pi(s) + X(s) M(s) + M^T(s) X^T(s) < 0$,
- iv) $(M^\perp(s))^T \Pi(s) M^\perp(s) < 0$.

Remark 2: Finsler's lemma [36] has been frequently utilized in previous researches. In [37], it is extended to the parameter-dependent systems for dealing with uncertain systems and contains proof of the extended Finsler's lemma. Lemma 4 is used to derive the inequality and understand the AZEA in the Theorems of this paper.

3. MAIN RESULTS

This paper considers the main results using Lemmas 1-4 in the NNTVDs. To represent the main results, block entry matrices, ϱ_i ($i = 1, 2, \dots, 18$) $\in \mathbb{R}^{18n \times n}$, are usually used to simply express matrices and vector states. For examples, $\varrho_5 = [0_{n \times 4n}, I_n, 0_{n \times 13n}]^T$ and $\varrho_{11} = [0_{n \times 10n}, I_n, 0_{n \times 7n}]^T$.

Some notations are explained below as

$$\begin{aligned} \Gamma_p &= \text{diag}\{\gamma_1^+, \dots, \gamma_n^+\}, \\ \Gamma_m &= \text{diag}\{\gamma_1^-, \dots, \gamma_n^-\}, \\ \xi(t) &= [\xi_1^T(t), \xi_2^T(t), \xi_3^T(t), \xi_4^T(t), \xi_5^T(t)]^T, \\ \xi_1(t) &= [\mathcal{X}^T(t), \mathcal{X}^T(t - \varphi(t)), \mathcal{X}^T(t - \varphi_M), \dot{\mathcal{X}}^T(t), \\ & \quad \dot{\mathcal{X}}^T(t - \varphi_M)]^T, \\ \xi_2(t) &= \left[\int_{t-\varphi(t)}^t \mathcal{X}^T(s) ds, \int_{t-\varphi_M}^{t-\varphi(t)} \mathcal{X}^T(s) ds, \right. \\ & \quad \left. \frac{1}{\varphi(t)} \int_{t-\varphi(t)}^t \int_s^t \mathcal{X}^T(u) du ds, \right. \\ & \quad \left. \frac{1}{\varphi_M - \varphi(t)} \int_{t-\varphi_M}^{t-\varphi(t)} \int_s^{t-\varphi(t)} \mathcal{X}^T(u) du ds \right]^T, \\ \xi_3(t) &= \left[\frac{1}{\varphi(t)} \int_{t-\varphi(t)}^t \mathcal{X}^T(s) ds, \right. \\ & \quad \left. \frac{1}{\varphi_M - \varphi(t)} \int_{t-\varphi_M}^{t-\varphi(t)} \mathcal{X}^T(s) ds, \right]^T, \\ \xi_4(t) &= \left[f^T(\mathcal{X}(t)), f^T(\mathcal{X}(t - \varphi(t))), f^T(\mathcal{X}(t - \varphi_M)) \right]^T, \\ \xi_5(t) &= \left[\int_{t-\varphi(t)}^t f^T(W_2 \mathcal{X}(s)) ds, \int_{t-\varphi_M}^{t-\varphi(t)} f^T(W_2 \mathcal{X}(s)) ds, \right. \\ & \quad \left. \frac{1}{\varphi(t)} \int_{t-\varphi(t)}^t \int_s^t f^T(W_2 \mathcal{X}(u)) du ds, \right. \\ & \quad \left. \frac{1}{\varphi_M - \varphi(t)} \int_{t-\varphi_M}^{t-\varphi(t)} \int_s^{t-\varphi(t)} f^T(W_2 \mathcal{X}(u)) du ds \right]^T, \\ \alpha(t) &= [\mathcal{X}^T(t), \dot{\mathcal{X}}^T(t), f^T(W_2 \mathcal{X}(t))]^T, \\ \zeta_1(t) &= \left[\mathcal{X}^T(t), \mathcal{X}^T(t - \varphi_M), \int_{t-\varphi_M}^t \mathcal{X}^T(s) ds, \right. \\ & \quad \left. \int_{t-\varphi_M}^t \int_s^t \mathcal{X}^T(u) du ds, \int_{t-\varphi_M}^t f^T(W_2 \mathcal{X}(s)) ds, \right. \\ & \quad \left. \int_{t-\varphi_M}^t \int_s^t f^T(W_2 \mathcal{X}(u)) du ds \right]^T, \\ \zeta_2(t, s) &= \left[\alpha^T(s), \int_s^t \alpha^T(u) du \right]^T, \\ \zeta_3(t, s) &= \left[\mathcal{X}^T(s), f^T(W_2 \mathcal{X}(s)), \int_s^t \alpha^T(u) du, \right. \\ & \quad \left. \mathcal{X}^T(t - \varphi_M) \right]^T, \end{aligned}$$

$$\begin{aligned}
Q_1^{aug} &= Q + \begin{bmatrix} 0_n & P_1 & 0_n \\ \star & 0_n & 0_n \\ \star & \star & 0_n \end{bmatrix}, \\
Q_2^{aug} &= Q + \begin{bmatrix} 0_n & P_2 & 0_n \\ \star & 0_n & 0_n \\ \star & \star & 0_n \end{bmatrix}, \\
\Sigma_{1[\varphi(t)]} &= \text{Sym} \{ \sigma_{11[\varphi(t)]} R \sigma_{12} \}, \\
\sigma_{11[\varphi(t)]} &= [\varrho_1, \varrho_3, \varrho_6 + \varrho_7, \varphi(t) \varrho_8 + (\varphi_M - \varphi(t))(\varrho_9 + \varrho_6), \\
&\quad \varrho_{15} + \varrho_{16} \varphi(t) \varrho_{17} + (\varphi_M - \varphi(t))(\varrho_{18} + \varrho_{15})], \\
\sigma_{12} &= [\varrho_4, \varrho_5, \varrho_1 - \varrho_3, \varphi_M \varrho_1 - (\varrho_6 + \varrho_7), \varrho_{12} - \varrho_{14}, \\
&\quad \varphi_M \varrho_{12} - (\varrho_{15} + \varrho_{16})]^T, \\
\Sigma_{2[\varphi(t)]} &= [\sigma_{21}, \varrho_0, \varrho_0, \varrho_0] N [\sigma_{21}, \varrho_0, \varrho_0, \varrho_0]^T \\
&\quad - \sigma_{22} N \sigma_{22}^T \\
&\quad + \text{Sym} \{ \sigma_{23[\varphi(t)]} N [\varrho_0, \varrho_0, \varrho_0, \sigma_{21}]^T \} \\
&\quad + \text{Sym} \{ (\varrho_{12} - \varrho_1 W_2^T \Gamma_m) \Lambda_1 W_2 \varrho_4^T \} \\
&\quad + \text{Sym} \{ (\varrho_1 W_2^T \Gamma_p - \varrho_{12}) \Delta_1 W_2 \varrho_4^T \} \\
&\quad + \text{Sym} \{ (\varrho_{14} - \varrho_3 W_2^T \Gamma_m) \Lambda_2 W_2 \varrho_5^T \} \\
&\quad + \text{Sym} \{ (\varrho_3 W_2^T \Gamma_p - \varrho_{14}) \Delta_2 W_2 \varrho_5^T \}, \\
\sigma_{21} &= [\varrho_1, \varrho_4, \varrho_{12}], \\
\sigma_{22} &= [\varrho_3, \varrho_5, \varrho_{14}, \varrho_6 + \varrho_7, \varrho_1 - \varrho_3, \varrho_{15} + \varrho_{16}], \\
\sigma_{23[\varphi(t)]} &= [\varrho_6 + \varrho_7, \varrho_1 - \varrho_3, \varrho_{15} + \varrho_{16}, \\
&\quad \varphi(t) \varrho_8 + (\varphi_M - \varphi(t))(\varrho_6 + \varrho_9), \\
&\quad \varphi_M \varrho_1 - (\varrho_6 + \varrho_7), \\
&\quad \varphi(t) \varrho_{17} + (\varphi_M - \varphi(t))(\varrho_{15} + \varrho_{18})], \\
\Sigma_{31} &= [\varrho_1, \varrho_{12}, \varrho_0, \varrho_0, \varrho_0, \varrho_3] G [\varrho_1, \varrho_{12}, \\
&\quad \varrho_0, \varrho_0, \varrho_0, \varrho_3]^T - (1 - \varphi_D) \sigma_{31} G \sigma_{31}^T, \\
\sigma_{31} &= [\varrho_2, \varrho_{13}, \varrho_6, \varrho_1 - \varrho_2, \varrho_{15}, \varrho_3], \\
\Sigma_{32[\varphi(t)]} &= \text{Sym} \{ \sigma_{32[\varphi(t)]} G [\varrho_0, \varrho_0, \varrho_1, \varrho_4, \varrho_{12}, \varrho_5]^T \}, \\
\sigma_{32[\varphi(t)]} &= [\varrho_6, \varrho_{15}, \varphi(t) \varrho_8, \varphi(t) \varrho_1 - \varrho_6, \varphi(t) \varrho_{17}, \\
&\quad \varphi(t) \varrho_3], \\
\Sigma_4 &= \varphi_M [\varrho_1, \varrho_4, \varrho_{12}] Q [\varrho_1, \varrho_4, \varrho_{12}]^T, \\
\Sigma_{41} &= \text{Sym} \{ S_1 v_1^T + U_1 v_3^T \}, \\
\Sigma_{42} &= \text{Sym} \{ S_2 v_2^T + U_2 v_4^T \}, \\
v_i &= [\varrho_{i+5}, \varrho_i - \varrho_{i+1}, \varrho_{i+14}], (i = 1, 2), \\
v_j &= [-\varrho_{j+3} + 2\varrho_{j+5}, \varrho_{j-2} + \varrho_{j-1} - 2\varrho_{j+7}, \\
&\quad -\varrho_{j+12} + 2\varrho_{j+14}], (j = 3, 4), \\
\Sigma_5 &= \varrho_1 P_1 \varrho_1^T - \varrho_2 (P_1 - P_2) \varrho_2^T - \varrho_3 P_2 \varrho_3^T, \\
\Sigma_{6[\varphi(t)]} &= \text{Sym} \{ (\varphi(t) \varrho_{10} - \varrho_6) \Psi_1 \} \\
&\quad + \text{Sym} \{ ((\varphi_M - \varphi(t)) \varrho_{11} - \varrho_7) \Psi_2 \}, \\
\Theta_{1[\varphi(t)]} &= - \sum_{i=1}^3 \text{Sym} \{ (\varrho_{11+i} - \varrho_i W_2^T \Gamma_m)
\end{aligned}$$

$$\begin{aligned}
&\times \left(\frac{\varphi(t)}{\varphi_M} J_i + \frac{\varphi_M - \varphi(t)}{\varphi_M} L_i \right) \\
&\times (\varrho_{11+i} - \varrho_i W_2^T \Gamma_p)^T \},
\end{aligned}$$

$$\begin{aligned}
\Theta_{2[\varphi(t)]} &= - \sum_{j=1}^2 \text{Sym} \{ (\varrho_{11+j} - \varrho_{12+j} \\
&\quad - (\varrho_j - \varrho_{1+j}) W_2^T \Gamma_m) \\
&\quad \times \left(\frac{\varphi(t)}{\varphi_M} J_{3+j} + \frac{\varphi_M - \varphi(t)}{\varphi_M} L_{3+j} \right) \\
&\quad \times (\varrho_{11+j} - \varrho_{12+j} - (\varrho_j - \varrho_{1+j}) W_2^T \Gamma_p)^T \}, \\
\Sigma_{[\varphi(t)]} &= \Sigma_{1[\varphi(t)]} + \Sigma_{2[\varphi(t)]} + \Sigma_{31} + \Sigma_{32[\varphi(t)]} + \Sigma_4 + \Sigma_{41} \\
&\quad + \Sigma_{42} + \Sigma_5 + \Sigma_{6[\varphi(t)]} + \Theta_{1[\varphi(t)]} + \Theta_{2[\varphi(t)]}, \\
\Pi &= -A \varrho_1^T - \varrho_4^T + W_0 \varrho_{12}^T + W_1 \varrho_{13}^T. \quad (4)
\end{aligned}$$

Now, the following theorem is given by the first main result.

Theorem 1: For given scalars φ_M and φ_D , system (1) is asymptotically stable for $0 \leq \varphi(t) \leq \varphi_M$ and $\dot{\varphi}(t) \leq \varphi_D$, if there exist matrices $R, G, N \in \mathbb{S}_+^{6n}, Q \in \mathbb{S}_+^{3n}, \Lambda_i, \Delta_i \in \mathbb{D}_+^n, P_i \in \mathbb{S}^n, S_i, U_i \in \mathbb{R}^{18n \times 3n}, \Psi_i \in \mathbb{R}^{n \times 18n} (i = 1, 2), J_j \in \mathbb{D}_+^n, L_j \in \mathbb{D}_+^n (j = 1, 2, \dots, 5)$ satisfying the following LMIs

$$\begin{bmatrix} \Pi_{\perp}^T \Sigma_{[\varphi_M]} \Pi_{\perp} & \Pi_{\perp}^T \varphi_M [S_1, U_1] \\ \star & -\varphi_M \text{diag} \{ Q_1^{aug}, 3Q_1^{aug} \} \end{bmatrix} < 0, \quad (5)$$

$$\begin{bmatrix} \Pi_{\perp}^T \Sigma_{[0]} \Pi_{\perp} & \Pi_{\perp}^T \varphi_M [S_2, U_2] \\ \star & -\varphi_M \text{diag} \{ Q_2^{aug}, 3Q_2^{aug} \} \end{bmatrix} < 0, \quad (6)$$

$$Q_1^{aug} \geq 0, \quad (7)$$

$$Q_2^{aug} \geq 0. \quad (8)$$

Proof: Let us see the following LKFs:

$$V(t) = V_1 + V_2 + V_3 + V_4, \quad (9)$$

where

$$V_1 = \zeta_1^T(t) R \zeta_1(t),$$

$$\begin{aligned}
V_2 &= \int_{t-\varphi_M}^t \zeta_2^T(t, s) N \zeta_2(t, s) ds \\
&\quad + 2 \sum_{i=1}^n \left(\lambda_{1,i} \int_0^{W_{2,i} \varphi(t)} (f_i(s) - \gamma_i^- s) ds \right) \\
&\quad + 2 \sum_{i=1}^n \left(\delta_{1,i} \int_0^{W_{2,i} \varphi(t)} (\gamma_i^+ s - f_i(s)) ds \right) \\
&\quad + 2 \sum_{i=1}^n \left(\lambda_{2,i} \int_0^{W_{2,i} \varphi(t-\varphi_M)} (f_i(s) - \gamma_i^- s) ds \right) \\
&\quad + 2 \sum_{i=1}^n \left(\delta_{2,i} \int_0^{W_{2,i} \varphi(t-\varphi_M)} (\gamma_i^+ s - f_i(s)) ds \right),
\end{aligned}$$

$$V_3 = \int_{t-\varphi(t)}^t \zeta_3^T(t, s) G \zeta_3(t, s) ds,$$

$$V_4 = \int_{t-\varphi_M}^t \int_s^t \alpha^T(u) Q \alpha(u) du ds.$$

The time derivatives of V_1 and V_2 are written as

$$\dot{V}_1 = 2\zeta_1^T(t)R\dot{\zeta}_1(t) = \xi^T(t)\Sigma_1[\varphi(t)]\xi(t), \quad (10)$$

$$\begin{aligned} \dot{V}_2 = & \zeta_2^T(t,t)N\zeta_2(t,t) - \zeta_2^T(t,t-\varphi_M)N\zeta_2(t,t-\varphi_M) \\ & + 2 \left[\int_{t-\varphi_M}^t \alpha(s)ds \right]^T N \begin{bmatrix} 0_{3n \times 1} \\ \alpha(t) \end{bmatrix} \\ & + 2(f(W_2\kappa(t)) - \Gamma_m W_2\kappa(t))^T \Delta_1 W_2\dot{\kappa}(t) \\ & + 2(\Gamma_p W_2\kappa(t) - f(W_2\kappa(t)))^T \Delta_1 W_2\dot{\kappa}(t) \\ & + 2(f(W_2\kappa(t-\varphi_M)) - \Gamma_m W_2\kappa(t-\varphi_M))^T \\ & \times \Delta_2 W_2\dot{\kappa}(t-\varphi_M) \\ & + 2(\Gamma_p W_2\kappa(t-\varphi_M) - f(W_2\kappa(t-\varphi_M)))^T \\ & \times \Delta_2 W_2\dot{\kappa}(t-\varphi_M) \\ = & \xi^T(t)\Sigma_2[\varphi(t)]\xi(t). \end{aligned} \quad (11)$$

Calculating the upper bounds of the time derivatives of V_3 can lead to

$$\begin{aligned} \dot{V}_3 = & \zeta_3^T(t,t)G\zeta_3(t,t) \\ & - (1-\phi(t))\zeta_3^T(t,t-\varphi(t))G\zeta_3(t,t-\varphi(t)) \\ & + \int_{t-\varphi(t)}^t \frac{d}{dt} (\zeta_3^T(t,s)G\zeta_3(t,s))ds \\ \leq & \zeta_3^T(t,t)G\zeta_3(t,t) \\ & - (1-\varphi_D)\zeta_3^T(t,t-\varphi(t))G\zeta_3(t,t-\varphi(t)) \\ & + \int_{t-\varphi(t)}^t \frac{d}{dt} (\zeta_3^T(t,s)G\zeta_3(t,s))ds \\ = & \xi^T(t)(\Sigma_{31} + \Sigma_{32}[\varphi(t)])\xi(t), \end{aligned} \quad (12)$$

because of the condition $\phi(t) \leq \varphi_D$ in (2).

Two zero equalities are considered by the Newton-Leibniz formula as follows:

$$\begin{aligned} 0 = & \kappa^T(t)P_1\kappa(t) - \kappa^T(t-\varphi(t))P_1\kappa(t-\varphi(t)) \\ & - 2 \int_{t-\varphi(t)}^t \kappa^T(s)P_1\dot{\kappa}(s)ds, \end{aligned} \quad (13)$$

$$\begin{aligned} 0 = & \kappa^T(t-\varphi(t))P_2\kappa(t-\varphi(t)) \\ & - \kappa^T(t-\varphi_M)P_2\kappa(t-\varphi_M) \\ & - 2 \int_{t-\varphi_M}^{t-\varphi(t)} \kappa^T(s)P_2\dot{\kappa}(s)ds. \end{aligned} \quad (14)$$

The sum of (13) and (14) can lead to

$$\begin{aligned} 0 = & \xi^T(t)\Sigma_5\xi(t) - 2 \int_{t-\varphi(t)}^t \kappa^T(s)P_1\dot{\kappa}(s)ds \\ & - 2 \int_{t-\varphi_M}^{t-\varphi(t)} \kappa^T(s)P_2\dot{\kappa}(s)ds. \end{aligned} \quad (15)$$

Calculating the time derivatives of V_4 is

$$\begin{aligned} \dot{V}_4 = & \varphi_M \alpha^T(t)Q\alpha(t) - \int_{t-\varphi_M}^t \alpha^T(s)Q\alpha(s)ds \\ = & \xi^T(t)\Sigma_4\xi(t) - \int_{t-\varphi_M}^t \alpha^T(s)Q\alpha(s)ds. \end{aligned} \quad (16)$$

By adding (15) into (16), it can be

$$\begin{aligned} \dot{V}_4 = & \xi^T(t)(\Sigma_4 + \Sigma_5)\xi(t) - \int_{t-\varphi(t)}^t \alpha^T(s)Q_1^{aug}\alpha(s)ds \\ & - \int_{t-\varphi_M}^{t-\varphi(t)} \alpha^T(s)Q_2^{aug}\alpha(s)ds. \end{aligned} \quad (17)$$

Using the integral inequality suggested in Lemma 2, the integral terms in (17) each have the upper bounds as

$$\begin{aligned} & - \int_{t-\varphi(t)}^t \alpha^T(s)Q_1^{aug}\alpha(s)ds \\ \leq & Sym\{\xi^T(t)S_1\omega_1(t) + \xi^T(t)U_1\omega_2(t)\} \\ & + \varphi(t)\xi^T(t)Q_1\xi(t), \end{aligned} \quad (18)$$

$$\begin{aligned} & - \int_{t-\varphi_M}^{t-\varphi(t)} \alpha^T(s)Q_2^{aug}\alpha(s)ds \\ \leq & Sym\{\xi^T(t)S_2\omega_3(t) + \xi^T(t)U_2\omega_4(t)\} \\ & + (\varphi_M - \varphi(t))\xi^T(t)Q_2\xi(t), \end{aligned} \quad (19)$$

where

$$\begin{aligned} Q_1 = & S_1(Q_1^{aug})^{-1}S_1^T + \frac{1}{3}U_1(Q_1^{aug})^{-1}U_1^T, \\ Q_2 = & S_2(Q_2^{aug})^{-1}S_2^T + \frac{1}{3}U_2(Q_2^{aug})^{-1}U_2^T, \\ \omega_1(t) = & \int_{t-\varphi(t)}^t \alpha(s)ds = [\varrho_6, \varrho_1 - \varrho_2, \varrho_{15}]^T \xi(t), \\ \omega_2(t) = & - \int_{t-\varphi(t)}^t \alpha(s)ds + \frac{2}{\varphi(t)} \int_{t-\varphi(t)}^t \int_s^t \alpha(u)duds \\ = & [-\varrho_6 + 2\varrho_8, \varrho_1 + \varrho_2 - 2\varrho_{10}, -\varrho_{15} + 2\varrho_{17}]^T \xi(t), \\ \omega_3(t) = & \int_{t-\varphi_M}^{t-\varphi(t)} \alpha(s)ds = [\varrho_7, \varrho_2 - \varrho_3, \varrho_{16}]^T \xi(t), \\ \omega_4(t) = & - \int_{t-\varphi_M}^{t-\varphi(t)} \alpha(s)ds \\ & + \frac{2}{\varphi_M - \varphi(t)} \int_{t-\varphi_M}^{t-\varphi(t)} \int_s^{t-\varphi(t)} \alpha(u)duds \\ = & [-\varrho_7 + 2\varrho_9, \varrho_2 + \varrho_3 - 2\varrho_{11}, -\varrho_{16} + 2\varrho_{18}]^T \xi(t). \end{aligned}$$

The upper bound of $\dot{V}_4$ derived using (16)-(19) is as follows:

$$\begin{aligned} \dot{V}_4 \leq & \xi^T(t)(\Sigma_4 + \Sigma_{41} + \Sigma_{42} + \Sigma_5)\xi(t) \\ & + \xi^T(t)(\varphi(t)Q_1 + (\varphi_M - \varphi(t))Q_2)\xi(t). \end{aligned} \quad (20)$$

In the component association of $\xi(t)$, two zero equalities are derived

$$0 = \xi^T(t)Sym\{(\varphi(t)\varrho_{10} - \varrho_6)\Psi_1\}\xi(t), \quad (21)$$

$$0 = \xi^T(t)Sym\{((\varphi_M - \varphi(t))\varrho_{11} - \varrho_7)\Psi_2\}\xi(t), \quad (22)$$

where $\Psi_i \in \mathbb{R}^{n \times 18n}$ ($i = 1, 2$) are any matrices.

The sum of (21) and (22) can lead to

$$0 = \xi^T(t)\Sigma_6[\varphi(t)]\xi(t). \quad (23)$$

From Assumption 1 with $a = 0$, for $J_j, L_j \in \mathbb{D}_+^n$ ($j = 1, 2, 3$), the derived inequality is satisfied

$$\xi^T(t) \Theta_{1[\varphi(t)]} \xi(t) \geq 0, \quad (24)$$

and with $a \neq 0$, for $J_j, L_j \in \mathbb{D}_+^n$ ($j = 4, 5$), the derived inequality is also satisfied

$$\xi^T(t) \Theta_{2[\varphi(t)]} \xi(t) \geq 0. \quad (25)$$

The proof of the inequalities (24) and (25) was specifically explained in [13].

From (11)-(25), an upper bound of $\dot{V}(t)$ can be written as

$$\begin{aligned} \dot{V}(t) &\leq \xi^T(t) (\Sigma_{[\varphi(t)]} + \varphi(t) \mathcal{Q}_1) \xi(t) \\ &\quad + \xi^T(t) (\varphi_M - \varphi(t)) \mathcal{Q}_2 \xi(t). \end{aligned} \quad (26)$$

A stability condition in the system (1) can be

$$\Sigma_{[\varphi(t)]} + \varphi(t) \mathcal{Q}_1 + (\varphi_M - \varphi(t)) \mathcal{Q}_2 < 0, \quad (27)$$

subject to $0 = \Pi \xi(t)$.

From (i) and (ii) of Lemma 3, the following inequality satisfies

$$\Pi_{\perp}^T (\Sigma_{[\varphi(t)]} + \varphi(t) \mathcal{Q}_1 + (\varphi_M - \varphi(t)) \mathcal{Q}_2) \Pi_{\perp} < 0. \quad (28)$$

The inequality (28) is satisfied, if the following two inequalities hold

$$\Pi_{\perp}^T \left[\Sigma_{[\varphi_M]} + \varphi_M \mathcal{Q}_1 \right] \Pi_{\perp} < 0, \quad (29)$$

and

$$\Pi_{\perp}^T \left[\Sigma_{[0]} + \varphi_M \mathcal{Q}_2 \right] \Pi_{\perp} < 0, \quad (30)$$

since it relies on $\varphi(t)$ with the condition of $0 \leq \varphi(t) \leq \varphi_M$. Therefore, the stability conditions for GNNs (1) satisfy with (5)-(8), since the inequalities (29) and (30) can be developed by Schur's complements. Thus, it is asymptotically stable for $0 \leq \varphi(t) \leq \varphi_M$ and $\dot{\varphi}(t) \leq \varphi_D$. This completes our proof. $\square$

Remark 3: It was proved that inserting inequalities containing the activation function condition (3) provides enhanced stability criteria for GNNs in [7]. Like [25], $\frac{\varphi(t)}{\varphi_M} J_i + \frac{\varphi_M - \varphi(t)}{\varphi_M} L_i$ ($i = 1, \dots, 5$) are always positive definite since the time-varying delay $0 \leq \varphi(t) \leq \varphi_M$ relies on $\varphi(t)$.

To represent the following Theorem 2, $\tilde{\varrho}_i$ ($i = 1, \dots, 6$) $\in \mathbb{R}^{6n \times n}$, are used to simply express matrices and vector states. If $N > 0$ satisfies, the lower bound of $V(t)$ follows as

$$V(t) > \zeta_1^T(t) R \zeta_1(t) + \int_{t-\varphi_M}^t \zeta_2^T(t, s) N \zeta_2(t, s) ds. \quad (31)$$

By Lemma 1, the lower bound of $\int_{t-\varphi_M}^t \zeta_2^T(t, s) N \zeta_2(t, s) ds$ can be easily obtained as

$$\begin{aligned} &\int_{t-\varphi_M}^t \zeta_2^T(t, s) N \zeta_2(t, s) ds \\ &\geq \frac{1}{\varphi_M} \left(\int_{t-\varphi_M}^t \zeta_2(t, s) ds \right)^T N \left(\int_{t-\varphi_M}^t \zeta_2(t, s) ds \right) \\ &= \frac{1}{\varphi_M} \zeta_1^T(t) [\tilde{\varrho}_3, \tilde{\varrho}_1 - \tilde{\varrho}_2, \tilde{\varrho}_5, \tilde{\varrho}_4, \varphi_M \tilde{\varrho}_1 - \tilde{\varrho}_3, \tilde{\varrho}_6] N \\ &\quad \times [\tilde{\varrho}_3, \tilde{\varrho}_1 - \tilde{\varrho}_2, \tilde{\varrho}_5, \tilde{\varrho}_4, \varphi_M \tilde{\varrho}_1 - \tilde{\varrho}_3, \tilde{\varrho}_6]^T \zeta_1(t). \end{aligned} \quad (32)$$

Therefore, if the following inequality holds,

$$\begin{aligned} 0 &< R + \frac{1}{\varphi_M} [\tilde{\varrho}_3, \tilde{\varrho}_1 - \tilde{\varrho}_2, \tilde{\varrho}_5, \tilde{\varrho}_4, \varphi_M \tilde{\varrho}_1 - \tilde{\varrho}_3, \tilde{\varrho}_6] \\ &\quad \times N [\tilde{\varrho}_3, \tilde{\varrho}_1 - \tilde{\varrho}_2, \tilde{\varrho}_5, \tilde{\varrho}_4, \varphi_M \tilde{\varrho}_1 - \tilde{\varrho}_3, \tilde{\varrho}_6]^T, \end{aligned} \quad (33)$$

then the lower bound of $V(t)$ can be guaranteed to be positive.

The following Theorem 2 shows the effectiveness of positiveness based on Theorem 1.

Theorem 2: For given scalars φ_M and φ_D , system (1) is asymptotically stable for $0 \leq \varphi(t) \leq \varphi_M$ and $\dot{\varphi}(t) \leq \varphi_D$, if there exist matrices $R \in \mathbb{S}^{6n}$, $G, N \in \mathbb{S}_+^{6n}$, $Q \in \mathbb{S}_+^{3n}$, $\Lambda_i, \Delta_i \in \mathbb{D}_+^n$, $P_i \in \mathbb{S}^n$, $S_i, U_i \in \mathbb{R}^{18n \times 3n}$, $\Psi_i \in \mathbb{R}^{n \times 18n}$ ($i = 1, 2$), $J_j \in \mathbb{D}_+^n$, $L_j \in \mathbb{D}_+^n$ ($j = 1, 2, \dots, 5$) satisfying the LMIs (5)-(8), and (33).

Proof: The augmented LKFs are the same in Theorem 1 (9) and has the lower bound $V(t)$ like (31). The other proof and notations are the same as the previous theorem, so it is omitted. $\square$

Remark 4: The lower bound of $V(t)$ can be held by integrating some functionals of $V(t)$ in Theorem 1. The positiveness of the matrix R replaces from $R > 0$ to (33). The proposed lower bound of $V(t)$ guarantees less conservatism in stability analysis.

The following Theorem 3 shows an improved result than the results of Theorem 2.

Theorem 3: For given scalars φ_M and φ_D , system (1) is asymptotically stable for $0 \leq \varphi(t) \leq \varphi_M$ and $\dot{\varphi}(t) \leq \varphi_D$, if there exist matrices $R \in \mathbb{S}^{6n}$, $G, N \in \mathbb{S}_+^{6n}$, $Q \in \mathbb{S}_+^{3n}$, $\Lambda_i, \Delta_i \in \mathbb{D}_+^n$, $P_i \in \mathbb{S}^n$, $S_i, U_i \in \mathbb{R}^{18n \times 3n}$ ($i = 1, 2$), $J_j \in \mathbb{D}_+^n$, $L_j \in \mathbb{D}_+^n$ ($j = 1, 2, \dots, 5$) satisfying (7), (8), (33), and the following LMIs

$$\begin{aligned} &\left[\begin{array}{cc} (\Pi_{[\varphi_M]})_{\perp}^T \bar{\Sigma}_{[\varphi_M]} (\Pi_{[\varphi_M]})_{\perp} & (\Pi_{[\varphi_M]})_{\perp}^T \varphi_M [S_1, U_1] \\ \star & -\varphi_M \text{diag}\{Q_1^{aug}, 3Q_1^{aug}\} \end{array} \right] \\ &< 0, \end{aligned} \quad (34)$$

$$\begin{aligned} &\left[\begin{array}{cc} (\Pi_{[0]})_{\perp}^T \bar{\Sigma}_{[0]} (\Pi_{[0]})_{\perp} & (\Pi_{[0]})_{\perp}^T \varphi_M [S_2, U_2] \\ \star & -\varphi_M \text{diag}\{Q_2^{aug}, 3Q_2^{aug}\} \end{array} \right] \\ &< 0, \end{aligned} \quad (35)$$

where $\bar{\Sigma}_{[\varphi(t)]} = \Sigma_{[\varphi(t)]} - \Sigma_{6[\varphi(t)]}$ and

$$\Pi_{[\varphi(t)]} = \begin{bmatrix} \Pi \\ \varphi(t)\varrho_{10}^T - \varrho_6^T \\ (\varphi_M - \varphi(t))\varrho_{11}^T - \varrho_7^T \end{bmatrix}.$$

The same notations are used, like Theorems 1 and 2.

Proof: Let us see the stability condition of the LKFs for a system (1). It can be written by

$$\begin{aligned} \dot{V}(t) &\leq \xi^T(t) [\bar{\Sigma}_{[\varphi(t)]} + \varphi(t)\mathcal{Q}_1 + (\varphi_M - \varphi(t))\mathcal{Q}_2] \xi(t) \\ &< 0, \\ \text{subject to } 0 &= \Pi_{[\varphi(t)]} \xi(t). \end{aligned} \quad (36)$$

Inequality (36) can be developed by Lemma 4, and it is equivalent to

$$\begin{aligned} &(\Pi_{[\varphi(t)]})_{\perp}^T [\bar{\Sigma}_{[\varphi(t)]} + \varphi(t)\mathcal{Q}_1 + (\varphi_M - \varphi(t))\mathcal{Q}_2] \\ &\quad \times (\Pi_{[\varphi(t)]})_{\perp} < 0. \end{aligned} \quad (37)$$

Inequality (37) is satisfied, if the following two inequalities hold

$$(\Pi_{[\varphi_M]})_{\perp}^T [\bar{\Sigma}_{[\varphi_M]} + \varphi_M\mathcal{Q}_1] (\Pi_{[\varphi_M]})_{\perp} < 0, \quad (38)$$

$$(\Pi_{[0]})_{\perp}^T [\bar{\Sigma}_{[0]} + \varphi_M\mathcal{Q}_2] (\Pi_{[0]})_{\perp} < 0, \quad (39)$$

since the above inequality relies on $\varphi(t)$ with the condition of $0 \leq \varphi(t) \leq \varphi_M$. Since the other proof is the same as Theorem 1, it can be omitted. $\square$

Remark 5: Zero equalities have some drawbacks, which increase the calculating costs due to the free variables $\Psi_i \in \mathbb{R}^{n \times 18n}$ ($i = 1, 2$). For solving these problems, the AZEA was proposed in [32]. All the zero equalities such that $0 = (-A\varrho_1^T - \varrho_4^T + W_0\varrho_{12}^T + W_1\varrho_{13}^T)\xi(t)$, $0 = (\varphi(t)\varrho_{10} - \varrho_6)\xi(t)$, $0 = ((\varphi_M - \varphi(t))\varrho_{11} - \varrho_7)\xi(t)$ in Theorem 3 are incorporated in $0 = \Pi_{[\varphi(t)]}\xi(t)$. By utilizing the AZEA, Lemma 4 was applied to (36). Consequently, it is possible not only to eliminate the free variables $\Psi_i \in \mathbb{R}^{n \times 18n}$ ($i = 1, 2$), but also to guarantee less conservatism in stability criteria with improved maximum upper bounds of time-varying delays.

Remark 6: The proposed LKFs in previous researches such as [9, 23] have not focused on and been not commonly used the activation function-based terms in NNTVDs. By including the activation function-based terms such as $\int_{t-\varphi_M}^t f(W_2\varpi(s))ds$ and $\int_{t-\varphi_M}^t \int_s^t f(W_2\varpi(u))duds$, it can be expected the improved delay bounds than previous papers. To prove the effectiveness of the proposed LKFs above, it is suggested the other LKFs except for the vectors of the activation function-based terms in Corollary 1. By comparing the results in three examples, the effectiveness of the proposed LKFs will be verified.

The following Corollary 1 shows for comparison how much the activation function-based terms are improved results. In order not to duplicate the notation of the preceding content, block entry matrices are declared such that $\hat{\varrho}_i = [0_{n \times (i-1)n}, I_n, 0_{n \times (14-i)n}]^T$ ($i = 1, 2, \dots, 14$) and $\hat{\varrho}_j = [0_{n \times (j-1)n}, I_n, 0_{n \times (4-j)n}]^T$ ($j = 1, 2, 3, 4$).

The notations for understanding the main results in Corollary 1 are defined as

$$\begin{aligned} \hat{\xi}(t) &= [\xi_1^T(t), \xi_2^T(t), \xi_3^T(t), \xi_4^T(t)]^T, \\ \hat{\alpha}(t) &= [\varpi^T(t), \varpi^T(t)]^T, \\ \hat{\xi}_1(t) &= [\varpi^T(t), \varpi^T(t - \varphi_M), \\ &\quad \int_{t-\varphi_M}^t \varpi^T(s)ds, \int_{t-\varphi_M}^t \int_s^t \varpi^T(u)duds]^T, \\ \hat{\xi}_2(t, s) &= [\hat{\alpha}^T(s), \int_s^t \hat{\alpha}^T(u)du]^T, \\ \hat{\xi}_3(t, s) &= [\varpi^T(s), \int_s^t \hat{\alpha}^T(u)du, \varpi^T(t - \varphi_M)]^T, \\ \hat{Q}_1^{aug} &= \hat{Q} + \begin{bmatrix} 0_n & P_1 \\ \star & 0_n \end{bmatrix}, \quad \hat{Q}_2^{aug} = \hat{Q} + \begin{bmatrix} 0_n & P_2 \\ \star & 0_n \end{bmatrix}, \\ \hat{\Sigma}_{1[\varphi(t)]} &= Sym\{\hat{\sigma}_{11[\varphi(t)]}\hat{R}\hat{\sigma}_{12}\}, \\ \hat{\sigma}_{11[\varphi(t)]} &= [\hat{\varrho}_1, \hat{\varrho}_3, \hat{\varrho}_6 + \hat{\varrho}_7, \\ &\quad \varphi(t)\hat{\varrho}_8 + (\varphi_M - \varphi(t))(\hat{\varrho}_9 + \hat{\varrho}_6)], \\ \hat{\sigma}_{12} &= [\hat{\varrho}_4, \hat{\varrho}_5, \hat{\varrho}_1 - \hat{\varrho}_3, \varphi_M\hat{\varrho}_1 - (\hat{\varrho}_6 + \hat{\varrho}_7)]^T, \\ \hat{\Sigma}_{2[\varphi(t)]} &= [\hat{\sigma}_{21}, \varrho_0, \varrho_0, \varrho_0]\hat{N}[\hat{\sigma}_{21}, \varrho_0, \varrho_0, \varrho_0]^T \\ &\quad - \sigma_{22}\hat{N}\hat{\sigma}_{22}^T \\ &\quad + Sym\{\hat{\sigma}_{23[\varphi(t)]}\hat{N}[\varrho_0, \varrho_0, \varrho_0, \hat{\sigma}_{21}]^T\} \\ &\quad + Sym\{(\hat{\varrho}_{12} - \hat{\varrho}_1W_2^T\Gamma_m)\hat{\Lambda}_1W_2\hat{\varrho}_4^T\} \\ &\quad + Sym\{(\hat{\varrho}_1W_2^T\Gamma_p - \hat{\varrho}_{12})\hat{\Lambda}_1W_2\hat{\varrho}_4^T\} \\ &\quad + Sym\{(\hat{\varrho}_{14} - \hat{\varrho}_3W_2^T\Gamma_m)\hat{\Lambda}_2W_2\hat{\varrho}_5^T\} \\ &\quad + Sym\{(\hat{\varrho}_3W_2^T\Gamma_p - \hat{\varrho}_{14})\hat{\Lambda}_2W_2\hat{\varrho}_5^T\}, \\ \hat{\sigma}_{21} &= [\hat{\varrho}_1, \hat{\varrho}_4], \\ \hat{\sigma}_{22} &= [\hat{\varrho}_3, \hat{\varrho}_5, \hat{\varrho}_6 + \hat{\varrho}_7, \hat{\varrho}_1 - \hat{\varrho}_3], \\ \hat{\sigma}_{23[\varphi(t)]} &= [\hat{\varrho}_6 + \hat{\varrho}_7, \hat{\varrho}_1 - \hat{\varrho}_3, \\ &\quad \varphi(t)\hat{\varrho}_8 + (\varphi_M - \varphi(t))(\hat{\varrho}_6 + \hat{\varrho}_9)], \\ \hat{\Sigma}_{31} &= [\hat{\varrho}_1, \hat{\varrho}_0, \hat{\varrho}_0, \hat{\varrho}_3]\hat{G}[\hat{\varrho}_1, \hat{\varrho}_0, \hat{\varrho}_0, \hat{\varrho}_3]^T \\ &\quad - (1 - \varphi_D)\hat{\sigma}_{31}\hat{G}\hat{\sigma}_{31}^T, \\ \hat{\sigma}_{31} &= [\hat{\varrho}_2, \hat{\varrho}_6, \hat{\varrho}_1 - \hat{\varrho}_2, \hat{\varrho}_3], \\ \hat{\Sigma}_{32[\varphi(t)]} &= Sym\{\hat{\sigma}_{32[\varphi(t)]}\hat{G}[\hat{\varrho}_0, \hat{\varrho}_1, \hat{\varrho}_4, \hat{\varrho}_5]^T\}, \\ \hat{\sigma}_{32[\varphi(t)]} &= [\hat{\varrho}_6, \varphi(t)\hat{\varrho}_8, \varphi(t)\hat{\varrho}_1 - \hat{\varrho}_6, \varphi(t)\hat{\varrho}_3], \\ \hat{\Sigma}_4 &= \varphi_M[\hat{\varrho}_1, \hat{\varrho}_4]\hat{Q}[\hat{\varrho}_1, \hat{\varrho}_4]^T, \\ \hat{\Sigma}_{41} &= Sym\{\hat{S}_1\hat{V}_1^T + \hat{U}_1\hat{V}_3^T\}, \\ \hat{\Sigma}_{42} &= Sym\{\hat{S}_2\hat{V}_2^T + \hat{U}_2\hat{V}_4^T\}, \\ \hat{V}_i &= [\varrho_{i+5}, \varrho_i - \varrho_{i+1}] \quad (i = 1, 2), \end{aligned}$$

$$\begin{aligned}
\hat{v}_j &= [-\varrho_{j+3} + 2\varrho_{j+5}, \varrho_{j-2} + \varrho_{j-1} - 2\varrho_{j+7}], \\
&\quad (j = 3, 4), \\
\hat{\Sigma}_5 &= \hat{\varrho}_1 \hat{P}_1 \hat{\varrho}_1^T - \hat{\varrho}_2 (\hat{P}_1 - \hat{P}_2) \hat{\varrho}_2^T - \hat{\varrho}_3 \hat{P}_2 \hat{\varrho}_3^T, \\
\hat{\Theta}_{1[\varphi(t)]} &= -\sum_{i=1}^3 \text{Sym} \left\{ (\hat{\varrho}_{11+i} - \hat{\varrho}_i W_2^T \Gamma_m) \right. \\
&\quad \times \left(\frac{\varphi(t)}{\varphi_M} J_i + \frac{\varphi_M - \varphi(t)}{\varphi_M} L_i \right) \\
&\quad \times (\hat{\varrho}_{11+i} - \hat{\varrho}_i W_2^T \Gamma_p)^T \Big\}, \\
\hat{\Theta}_{2[\varphi(t)]} &= -\sum_{j=1}^2 \text{Sym} \left\{ (\hat{\varrho}_{11+j} - \hat{\varrho}_{12+j} \right. \\
&\quad - (\hat{\varrho}_j - \hat{\varrho}_{1+j}) W_2^T \Gamma_m) \\
&\quad \times \left(\frac{\varphi(t)}{\varphi_M} J_{3+j} + \frac{\varphi_M - \varphi(t)}{\varphi_M} L_{3+j} \right) \\
&\quad \times (\hat{\varrho}_{11+j} - \hat{\varrho}_{12+j} - (\hat{\varrho}_j - \hat{\varrho}_{1+j}) W_2^T \Gamma_p)^T \Big\}, \\
\hat{\Sigma}_{[\varphi(t)]} &= \Sigma_{1[\varphi(t)]} + \Sigma_{2[\varphi(t)]} + \Sigma_{31} + \Sigma_{32[\varphi(t)]} + \Sigma_4 + \Sigma_{41} \\
&\quad + \Sigma_{42} + \Sigma_5 + \Sigma_{6[\varphi(t)]} + \Theta_{1[\varphi(t)]} + \Theta_{2[\varphi(t)]}, \\
\hat{\Sigma}_{[\varphi(t)]} &= \hat{\Sigma}_{1[\varphi(t)]} + \hat{\Sigma}_{2[\varphi(t)]} + \hat{\Sigma}_{31} + \hat{\Sigma}_{32[\varphi(t)]} + \hat{\Sigma}_4 + \hat{\Sigma}_{41} \\
&\quad + \hat{\Sigma}_{42} + \hat{\Sigma}_5 + \hat{\Theta}_{1[\varphi(t)]} + \hat{\Theta}_{2[\varphi(t)]}, \\
\hat{\Pi} &= -A \hat{\varrho}_1^T - \hat{\varrho}_4^T + W_0 \hat{\varrho}_{12}^T + W_1 \hat{\varrho}_{13}^T. \tag{40}
\end{aligned}$$

Corollary 1: For given scalars φ_M and φ_D , system (1) is asymptotically stable for $0 \leq \varphi(t) \leq \varphi_M$ and $\dot{\varphi}(t) \leq \varphi_D$, if there exist matrices $\hat{R} \in \mathbb{S}^{4n}$, $\hat{G}, \hat{N} \in \mathbb{S}_+^{4n}$, $\hat{Q} \in \mathbb{S}_+^{2n}$, $\hat{\Lambda}_i, \hat{\Delta}_i \in \mathbb{D}_+^n$, $\hat{P}_i \in \mathbb{S}^n$, $\hat{S}_i, \hat{U}_i \in \mathbb{R}^{14n \times 2n}$ ($i = 1, 2$), $\hat{J}_j \in \mathbb{D}_+^n$, $\hat{L}_j \in \mathbb{D}_+^n$ ($j = 1, 2, \dots, 5$) satisfying the following LMIs

$$\begin{bmatrix} (\hat{\Pi}_{[\varphi_M]})^T \hat{\Sigma}_{[\varphi_M]} (\hat{\Pi}_{[\varphi_M]})^T & (\hat{\Pi}_{[\varphi_M]})^T \varphi_M [\hat{S}_1, \hat{U}_1] \\ \star & -\varphi_M \text{diag} \{ \hat{Q}_1^{aug}, 3\hat{Q}_1^{aug} \} \end{bmatrix} < 0, \tag{41}$$

$$\begin{bmatrix} (\hat{\Pi}_{[0]})^T \hat{\Sigma}_{[0]} (\hat{\Pi}_{[0]})^T & (\hat{\Pi}_{[0]})^T \varphi_M [\hat{S}_2, \hat{U}_2] \\ \star & -\varphi_M \text{diag} \{ \hat{Q}_2^{aug}, 3\hat{Q}_2^{aug} \} \end{bmatrix} < 0, \tag{42}$$

$$\hat{Q}_1^{aug} \geq 0, \tag{43}$$

$$\hat{Q}_2^{aug} \geq 0, \tag{44}$$

$$\hat{R} + \frac{1}{\varphi_M} \chi \hat{N} \chi^T > 0, \tag{45}$$

where

$$\begin{aligned}
\hat{\Pi}_{[\varphi(t)]} &= \begin{bmatrix} \hat{\Pi} \\ \varphi(t) \hat{\varrho}_{10}^T - \hat{\varrho}_6^T \\ (\varphi_M - \varphi(t)) \hat{\varrho}_{11}^T - \hat{\varrho}_7^T \end{bmatrix}, \\
\chi &= [\hat{\varrho}_3, \hat{\varrho}_1 - \hat{\varrho}_2, \hat{\varrho}_4, \varphi_M \hat{\varrho}_1 - \hat{\varrho}_3],
\end{aligned}$$

and the notations are defined as (40).

Proof: Let us see the following LKFs:

$$\hat{V}(t) = \hat{V}_1 + \hat{V}_2 + \hat{V}_3 + \hat{V}_4, \tag{46}$$

where

$$\begin{aligned}
\hat{V}_1 &= \hat{\xi}_1^T(t) \hat{R} \hat{\xi}_1(t), \\
\hat{V}_2 &= \int_{t-\varphi_M}^t \hat{\xi}_2^T(t, s) \hat{N} \hat{\xi}_2(t, s) ds \\
&\quad + 2 \sum_{i=1}^n \left(\hat{\lambda}_{1,i} \int_0^{W_{2,i} \varphi(t)} (f_i(s) - \gamma_i^- s) ds \right) \\
&\quad + 2 \sum_{i=1}^n \left(\hat{\delta}_{1,i} \int_0^{W_{2,i} \varphi(t)} (\gamma_i^+ s - f_i(s)) ds \right) \\
&\quad + 2 \sum_{i=1}^n \left(\hat{\lambda}_{2,i} \int_0^{W_{2,i} \varphi(t-\varphi_M)} (f_i(s) - \gamma_i^- s) ds \right) \\
&\quad + 2 \sum_{i=1}^n \left(\hat{\delta}_{2,i} \int_0^{W_{2,i} \varphi(t-\varphi_M)} (\gamma_i^+ s - f_i(s)) ds \right), \\
\hat{V}_3 &= \int_{t-\varphi(t)}^t \hat{\xi}_3^T(t, s) \hat{G} \hat{\xi}_3(t, s) ds, \\
\hat{V}_4 &= \int_{t-\varphi_M}^t \int_s^t \hat{\alpha}^T(u) \hat{Q} \hat{\alpha}(u) du ds.
\end{aligned}$$

With the similar processes of Theorems 1-3, one can easily obtain the following condition

$$\begin{aligned}
\dot{\hat{V}}(t) &\leq \hat{\xi}^T(t) (\hat{\Sigma}_{[\varphi(t)]} + \varphi(t) \mathcal{Q}_3 + (\varphi_M - \varphi(t)) \mathcal{Q}_4) \hat{\xi}(t) \\
&< 0, \\
\text{subject to } 0 &= \hat{\Pi}_{[\varphi(t)]} \hat{\xi}(t), \tag{47}
\end{aligned}$$

and

$$\hat{V}(t) > \hat{\xi}_1^T(t) \hat{R} \hat{\xi}_1(t) + \int_{t-\varphi_M}^t \hat{\xi}_2^T(t, s) \hat{N} \hat{\xi}_2(t, s) ds, \tag{48}$$

where $\mathcal{Q}_3 = \hat{S}_1 (\hat{Q}_1^{aug})^{-1} \hat{S}_1^T + \frac{1}{3} \hat{U}_1 (\hat{Q}_1^{aug})^{-1} \hat{U}_1^T$ and $\mathcal{Q}_4 = \hat{S}_2 (\hat{Q}_2^{aug})^{-1} \hat{S}_2^T + \frac{1}{3} \hat{U}_2 (\hat{Q}_2^{aug})^{-1} \hat{U}_2^T$. The other process is similar to the proofs of Theorems 1-3, so it is omitted. $\square$

4. NUMERICAL EXAMPLES

Three examples show that the proposed methods can be effective and provide less conservative results in this paper.

Example 1: Let us consider the system (1)

$$\begin{aligned}
A &= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, W_0 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, W_1 = \begin{bmatrix} 0.88 & 1 \\ 1 & 1 \end{bmatrix}, \\
W_2 &= I_2, \Gamma_p = \text{diag}\{0.4, 0.8\}, \Gamma_m = O_2.
\end{aligned}$$

In Table 1, the maximum delay bounds are shown for different values φ_D . Table 1 includes some recent results such as [8,9,21,22,38,39]. By comparing the results, it is explicitly verified that the proposed methods are enhanced. When $\varphi_D = 0.8$, the upper bound value is smaller than the result in [9]. Through the comparison of Theorem 3 and Corollary 1, the proposed LKFs are more effective

Table 1. φ_M with different φ_D (Example 1).

φ_D	0.8	0.9	Unknown
[8]	6.7001	4.0747	3.6066
[21]	6.7186	3.9623	3.6654
[23]	7.1718	4.2979	3.7615
[22]	7.2033	4.0663	3.6876
[38]	7.2202	4.2568	3.7484
[39]	8.0373	4.4659	3.7426
[9]	9.3998	4.6035	3.8840
Theorem 1	6.8886	4.2175	3.6353
Theorem 2	7.0061	4.2954	3.7183
Theorem 3	9.3612	5.1996	4.2019
Corollary 1	7.8554	4.7847	4.2018

with less conservatism. It can have a meaning of the comparisons since others of φ_D are larger. Lastly, Theorem 3 increases the maximum delay bounds and decreases the free variables $\Psi_i \in \mathbb{R}^{n \times 18n}$ ($i = 1, 2$) by utilizing the AZEA compared to Theorem 2.

Example 2: Let us consider the system (1)

$$A = \begin{bmatrix} 1.5 & 0 \\ 0 & 0.7 \end{bmatrix}, W_0 = \begin{bmatrix} 0.0503 & 0.0454 \\ 0.0987 & 0.2075 \end{bmatrix},$$

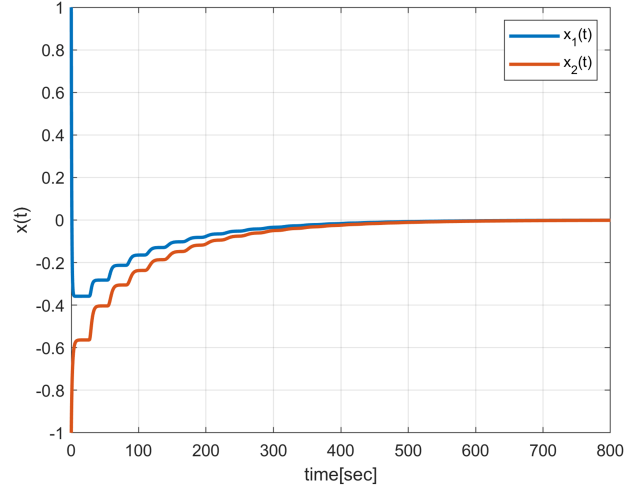
$$W_1 = \begin{bmatrix} 0.2381 & 0.9320 \\ 0.0388 & 0.5062 \end{bmatrix},$$

$$W_2 = I_2, \Gamma_p = \text{diag}\{0.3, 0.8\}, \Gamma_m = 0_2.$$

When the values of φ_D are 0.40, 0.45, 0.50, and 0.55, the maximum delay bounds are recorded in Table 2. It is explicitly verified that the proposed methods are enhanced than the previous literatures [8,9,21,22,38,39]. To check the effectiveness of the results for $\varphi_D = 0.40$ and $\varphi_M = 32.3833$, the state trajectories of Example 2 with time-varying delay $\varphi(t) = 0.3 \sin^2(t) + 32.0833$, activation function $f(\varkappa(t)) = [0.3 \tanh(\varkappa_1(t)), 0.8 \tanh(\varkappa_2(t))]^T$ and initial condition $\varkappa(0) = [1, -1]^T$ are shown in Fig. 1. It is proved that the system (1) with system parameters in Ex-

Table 2. φ_M with different φ_D (Example 2).

φ_D	0.40	0.45	0.50	0.55
[8]	17.2697	12.0698	10.2903	9.3879
[21]	18.8262	12.8039	10.6230	9.5038
[38]	20.1550	14.2404	12.0263	10.8213
[23]	20.8249	14.0082	11.5834	10.2690
[22]	21.0344	14.4707	11.9667	10.6439
[9]	27.7002	19.5050	16.4611	14.9900
[39]	27.7404	18.7296	15.8002	14.1240
Theorem 1	17.2194	12.2222	10.5582	9.7551
Theorem 2	17.5781	12.5651	10.8613	10.0089
Theorem 3	32.3833	22.5316	19.5912	18.3621
Corollary 1	31.4262	21.2865	17.7803	16.1689

Fig. 1. State trajectories of system (1) with $\varphi(t) = 0.3 \sin^2(t) + 32.0833$ (Example 2).

ample 2 is asymptotically stable since the state trajectories of Fig. 1 converge to zero as time goes to infinity.

Example 3: Let us consider the system (1)

$$A = \text{diag}\{7.3458, 6.9987, 5.5949\}, W_0 = 0_3,$$

$$W_1 = I_3, W_2 = \begin{bmatrix} 13.6014 & -2.9616 & -0.6936 \\ 7.4736 & 21.6810 & 3.2100 \\ 0.7290 & -2.6334 & -20.1300 \end{bmatrix},$$

$$\Gamma_p = \text{diag}\{0.3680, 0.1795, 0.2876\}, \Gamma_m = 0_3.$$

According to different φ_D , it is confirmed that the maximum delay bounds are listed in Table 3. It includes some recent results [8,9,21-23]. By comparing the results in Table 3, the proposed results guarantee less conservative regions than existing literatures. It is verified that the AZEA is effective technique for increasing the upper bounds delay. When $\varphi_D = 0.1$ and $\varphi_M = 1.1814$ are given, the state trajectories of Example 3 with time-varying delay $\varphi(t) = 0.1 |\sin(t)| + 1.0814$, activation function $f(\varkappa(t)) = [0.3680 \tanh(\varkappa_1(t)), 0.1795 \tanh(\varkappa_2(t)), 0.2876 \tanh(\varkappa_3(t))]^T$ and initial condition $\varkappa(0) = [2, -3,$

Table 3. φ_M with different φ_D (Example 3).

φ_D	0.1	0.5	0.9
[8]	1.1194	0.4599	0.4027
[23]	1.1205	0.4614	0.3963
[21]	1.1365	0.4678	-
[22]	1.1366	0.4887	0.4324
[9]	1.1705	0.5437	0.4620
Theorem 1	1.1201	0.4614	0.4031
Theorem 2	1.1202	0.4614	0.4032
Theorem 3	1.1814	0.6687	0.5938
Corollary 1	1.1758	0.6668	0.5938

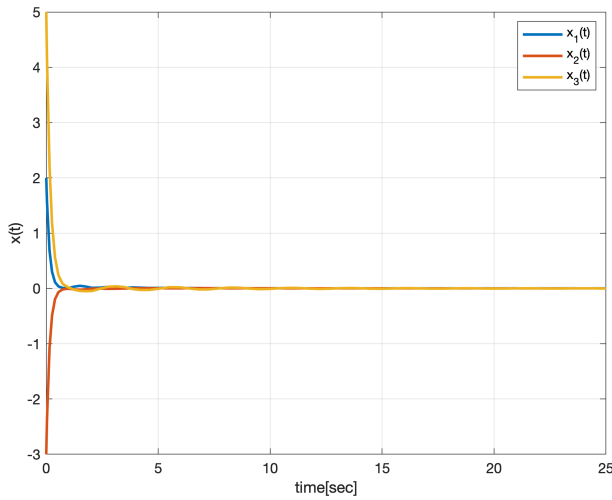


Fig. 2. State trajectories of system (1) with $\varphi(t) = 0.1|\sin(t)| + 1.0814$ (Example 3).

$5]^T$ are shown in Fig. 2. The system (1) with system parameters in Example 3 is asymptotically stable since the state trajectories of Fig. 2 are stable.

5. CONCLUSIONS

In this paper, enhanced delay-dependent stability criteria for GNNs are proposed. The asymptotic stability conditions are proposed by constructing the augmented LKFs obtaining the activation function-based terms and utilizing GFWMI. They show the distinction for constructing the proposed LKFs in Theorem 1. In particular, the combination with several mathematical techniques such as integral inequalities and positiveness shows that the proposed LKFs are effective in widening the maximum delay bounds. Based on Theorem 2, the AEZA in [32] is applied to a more relaxed criterion, eliminating free-variables caused zero equality and expanding the maximum delay bounds. Three numerical examples are used to verify the validity and superiority than previous literature. After the predicting the maximum delay bounds, the results are checked by the graph as shown in Figs. 1 and 2. Based on the proposed method, studies on the controller design for various systems such as ball & beams and some practical applications will be conducted in the authors' future works.

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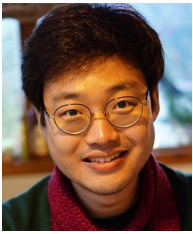
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