

A Study on Stability Analysis and The Estimation of The Reachable Set for Various Neural Networks Systems

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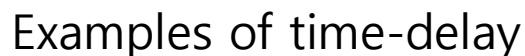
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1. Introduction

- Time-delay
- Neural networks
- Markovian jump process
- Linear matrix inequality(LMI)
- Motivation

- Time-delay
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- The cause of instability and poor performance.
- Communication, Sensing and electronic systems and so on.

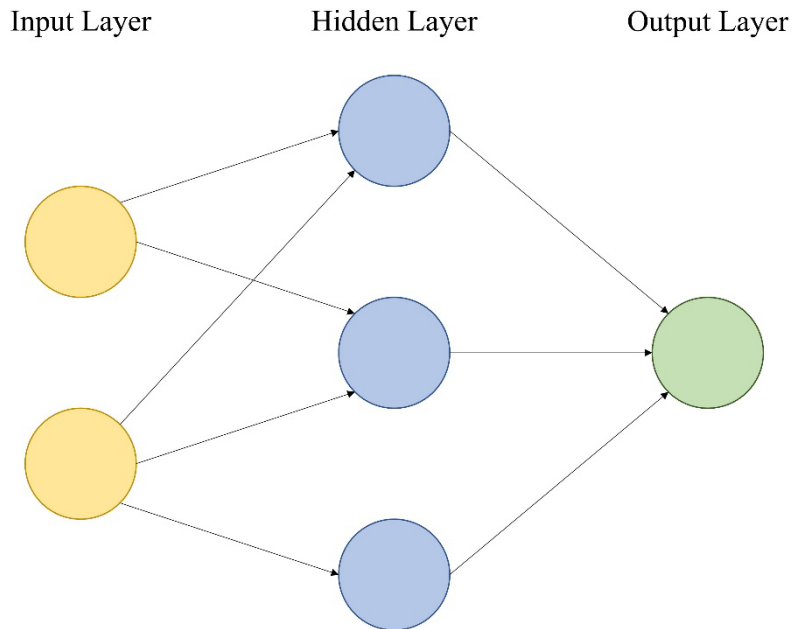


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1. Introduction

- Time-delay
- **Neural networks**
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What is neural networks?



Neural networks system consists of several layers such as input, hidden, and output.

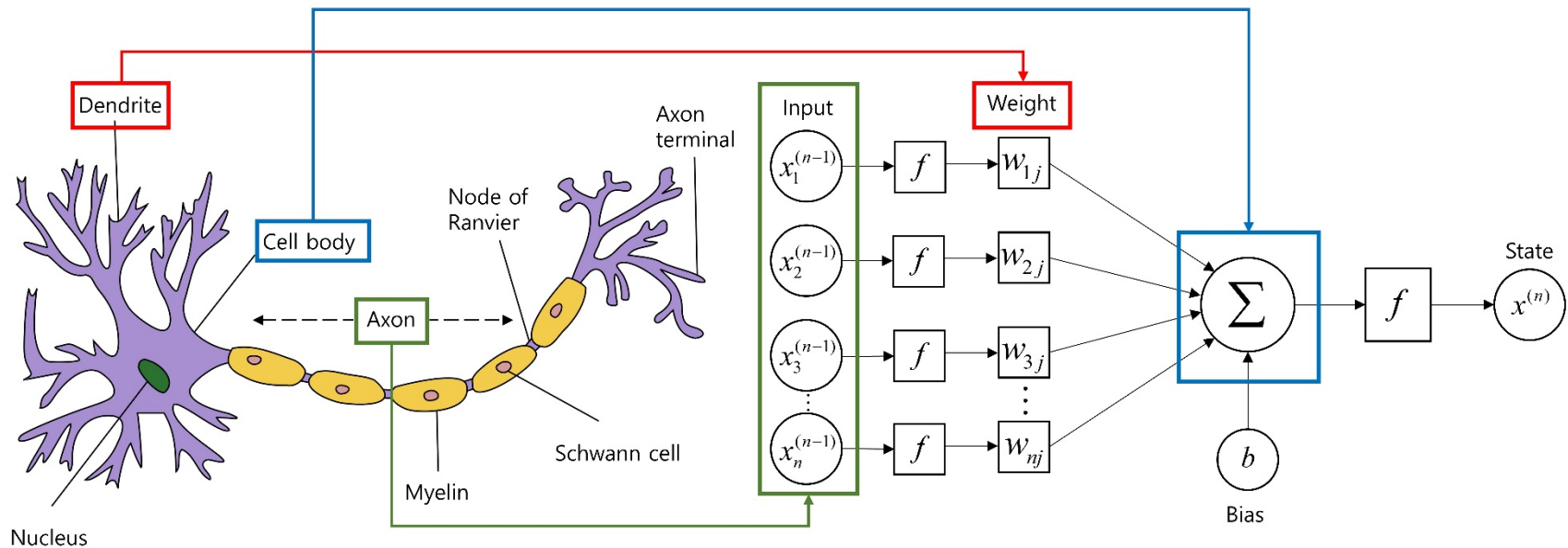
Each layer creates connections about some information.

These connections of layers are called the activation functions.

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How is the activation function structured?



$x_1 \dots x_n$ denote the neuron state and they have each weights.

These information of the neurons summarizes a function called by the activation function.

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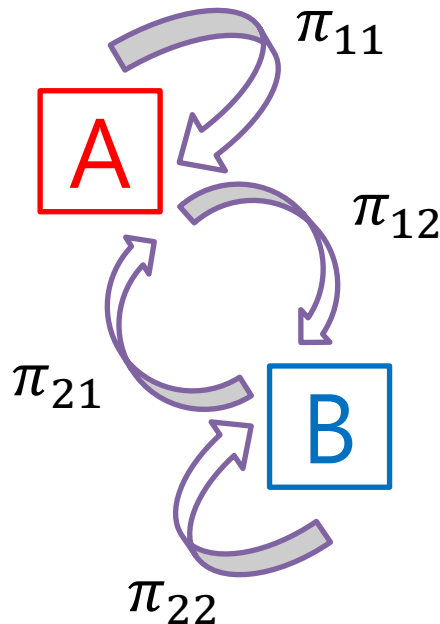
Markovian jump process

-The properties of nonlinear system
-Various natural phenomena



-Random change for a system

State diagram



The transition matrix is

$$P = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix}$$

with

$$S = \{1, 2\}.$$

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- Time-delay
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- **Linear matrix inequality(LMI)**
- Motivation

Linear matrix inequality (LMI)

- Given any matrices A , B and symmetric matrices P , Q , R

- Examples of LMI

$$A^T P + P A < 0,$$

$$\begin{bmatrix} A^T P + P A & B Q \\ Q B^T & -R \end{bmatrix} < 0.$$

LMI can be easily solved by using **Matlab LMI toolbox**.

1. Introduction

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Motivation

- Neural networks system is continuously considered in various fields.
- Time delay frequently occurs due to the finite switching speed of amplifiers and the communication among neurons in neural networks.
- Problems such as oscillations or destruction cause poor performances and instability.
- The study of time delay is in the spotlight in various engineering and medicine fields, and many researchers are interested in the neural network with time-varying delays.

2. Preliminaries

- Lyapunov-Krasovskii's theorem
- Neural networks system
- Markovian jump process
- Fact and lemmas
- Preview of the systems

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Lyapunov stability

Consider the Lyapunov functional as

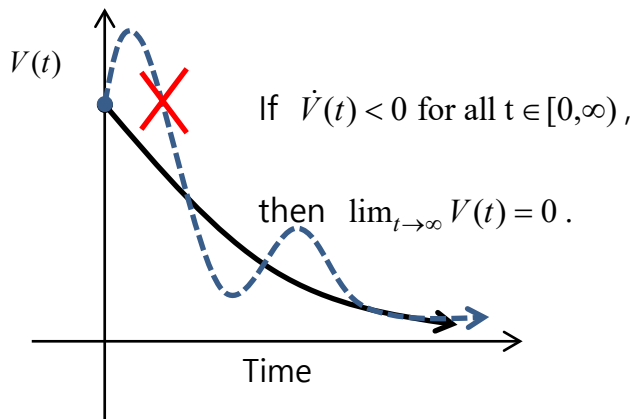
$$V(t) = x^T(t)Px(t), P > 0,$$

then,

$$V(t) \geq 0, \text{ for all } x(t).$$

For example, if $P = I$, then

$$V(t) = x_1^2(t) + x_2^2(t) + \dots + x_n^2(t).$$



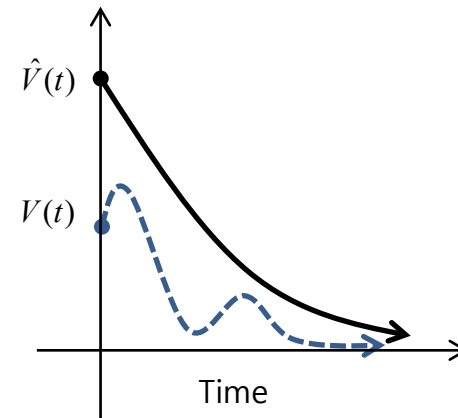
Lyapunov-Krasovskii's theorem

$$\hat{V}(t) = \underbrace{x^T(t)Px(t)}_{V(t)} + V_2(t) + V_3(t) + \dots + V_n(t)$$

where

$$V(t) \geq 0, V_2(t) \geq 0, \dots, V_n(t) \geq 0.$$

$$\dot{\hat{V}}(t) < 0 \text{ guarantees } \lim_{t \rightarrow \infty} V(t) = 0 \text{ (} \hat{V}(t) \geq V(t) \text{)}.$$



- It has less conservatism.

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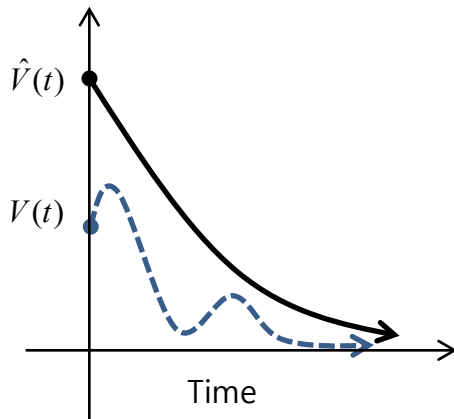
Lyapunov-Krasovskii's theorem

$$\hat{V}(t) = \underbrace{x^T(t)Px(t)}_{V(t)} + V_2(t) + V_3(t) + \dots + V_n(t)$$

where

$$V(t) \geq 0, \quad V_2(t) \geq 0, \quad \dots \quad V_n(t) \geq 0.$$

$$\dot{\hat{V}}(t) < 0 \text{ guarantees } \lim_{t \rightarrow \infty} V(t) = 0 \quad (\hat{V}(t) \geq V(t)).$$



- It has less conservatism.

Augmented

Lyapunov-Krasovskii's theorem

$$\hat{V}(t) = \underbrace{\begin{bmatrix} x(t) \\ \dot{x}(t) \\ \vdots \\ \int_{t-h_M}^t f(s)ds \end{bmatrix}^T P \begin{bmatrix} x(t) \\ \dot{x}(t) \\ \vdots \\ \int_{t-h_M}^t f(s)ds \end{bmatrix}}_{V(t)} + V_2(t) + V_3(t) + \dots + V_n(t)$$

where

$$V(t) \geq 0, \quad V_2(t) \geq 0, \quad \dots \quad V_n(t) \geq 0.$$

$$\dot{\hat{V}}(t) < 0 \text{ guarantees } \lim_{t \rightarrow \infty} V(t) = 0 \quad (\hat{V}(t) \geq V(t)).$$

2. Preliminaries

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Neural networks system with time-varying delays

Commonly used neural networks system with time-varying delays follows as

$$\dot{x}(t) = -Ax(t) + W_0 f(W_2 x(t)) + W_1 f(W_2 x(t - h(t))), \quad \forall t > 0,$$

where

$x(t)$ is the neuron state and $f(\cdot)$ is the activation function.

A is a system matrix, and W_0, W_1, W_2 are the interconnection matrices between neurons.

$h(t)$ is a continuous function which means the time-varying delays satisfying

$$0 \leq h(t) \leq h_M, \quad \dot{h}(t) \leq h_D,$$

where h_M is a positive constant and h_D is any scalar.

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Markovian jump process

From the work [3], let $\mathcal{S} = \{1, 2, \dots, N\}$ be an index set. The Markovian process $\{r(t), t \geq 0\}$ with values in the finite set $\mathcal{S}$, describes the switching between the different modes and its evolution is governed by the following probability transitions :

$$P\{r_{t+\Delta t} = q \mid r_t = p\} = \begin{cases} \pi_{pq}\Delta t + o(\Delta t), & p \neq q \\ 1 + \pi_{pp}\Delta t + o(\Delta t), & p = q \end{cases}$$

where $\Delta t > 0$ and $\lim_{\Delta t \rightarrow 0} \frac{o(\Delta t)}{\Delta t} = 0$. $\pi_{pq} \geq 0$ is the transition probability rate from mode p to mode q at

time t and $\pi_{pp} = -\sum_{\substack{q=1, \\ q \neq p}}^N \pi_{pq}$.

[3] E. K. Boukas, Introduction to functional differential equations, Birkhauser, Canada, (2006) 6-7.

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Fact 2.1 (Schur complement) [1]

Given constant matrices $\Sigma_1, \Sigma_2, \Sigma_3$ with $\Sigma_1 = \Sigma_1^T$ and $0 < \Sigma_2 = \Sigma_2^T$, then $\Sigma_1 + \Sigma_3 \Sigma_2^{-1} \Sigma_3^T < 0$ if and only if

$$\begin{bmatrix} \Sigma_1 & \Sigma_3^T \\ \Sigma_3 & -\Sigma_2 \end{bmatrix} < 0 \quad \text{or} \quad \begin{bmatrix} -\Sigma_2 & \Sigma_3 \\ \Sigma_3^T & \Sigma_1 \end{bmatrix} < 0.$$

Lemma 2.1.1 (Jensen's inequality) [37]

For a continuous function $\{x(s) | s \in [a, b]\} \rightarrow \mathbb{R}^n$, if a positive definite matrix N satisfies, the following integral inequality holds:

$$\frac{1}{b-a} \int_a^b x^T(s) N x(s) ds \geq \left(\int_a^b x(s) ds \right)^T N \left(\int_a^b x(s) ds \right).$$

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Lemma 2.1.2 (A generalized free-weighting-matrix inequality) [38]

For a continuous function $\{x(s) | s \in [a, b]\} \rightarrow \mathbb{R}^n$, a given positive definite matrix Q , any matrices S and U with appropriate dimensions and the following integral inequality holds:

$$-\int_a^b x^T(s) Q x(s) ds \leq \text{Sym} \left\{ \omega_0^T S \omega_1 + \omega_0^T U \omega_2 \right\} + (b-a) \omega_0^T \left(S Q^{-1} S^T + \frac{1}{3} U Q^{-1} U^T \right) \omega_0,$$

where ω_0 is any vector, $\omega_1 = \int_a^b x(s) ds$, and $\omega_2 = -\int_a^b x(s) ds + \frac{2}{b-a} \int_a^b \int_s^b x(u) du ds$.

Lemma 2.1.3 (Finsler's Lemma) [40]

Let $\zeta \in \mathbb{R}^n$, $\Pi = \Pi^T \in \mathbb{R}^{n \times n}$, and $M \in \mathbb{R}^{n \times m}$. The following statements are equivalent follow:

(i) $\zeta^T \Pi \zeta < 0$, $\forall M \zeta = 0$, $\zeta \neq 0$,

(ii) $(M^\perp)^T \Pi M^\perp < 0$,

(iii) $\exists X \in \mathbb{R}^{m \times n}$: $\Pi + \text{Sym}\{XM\} < 0$.

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Lemma 2.1.4 [39]

For each $s \in S$, let $S \subseteq \mathbb{R}^d$, $\Pi = \Pi^T \in \mathbb{R}^{n \times n}$, and $M \in \mathbb{R}^{m \times n}$. The following statements of equivalent follow:

- (i) $x^T \Pi x < 0$, $\forall Mx = 0$, $\zeta \neq 0$,
- (ii) $(\exists \mu(s) \in \mathbb{R}) : \Pi(s) - \mu(s) M^T(s) M(s) < 0$,
- (iii) If there exists $X(s) \in \mathbb{R}^{n \times m}$, $\Pi(s) + \text{Sym}\{X(s) M(s)\} < 0$,
- (iv) $(M^\perp(s))^T \Pi(s) M^\perp(s) < 0$.

Lemma 2.1.5 [47]

Let $V(x(t))$ be a differentiable Lyapunov function for system with $V(x(0)) = 0$ and external inputs $\omega(t)$ satisfying $\omega^T(t) \omega(t) \leq \omega_M^2$. If there exists scalars $\alpha > 0$ and $\beta > 0$ such that

$$\dot{V}(x(t)) + \alpha V(x(t)) - \beta \omega^T(t) \omega(t) \leq 0, \text{ then } V(x(t)) \leq \frac{\beta}{\alpha} \omega_M^2.$$

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- Lyapunov-Krasovskii's theorem
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- **Preview of the systems**

Preview of the systems

- A generalized neural networks system with time-varying delay

$$\dot{x}(t) = -Ax(t) + W_0 f(W_2 x(t)) + W_1 f(W_2 f(t - h(t))), \forall t > 0.$$

Section 3.1

- Markovian jump neural networks system with time-varying delays

$$\dot{x}(t) = -A(r_t)x(t) + W_0(r_t)f(W_1 x(t)) + W_2(r_t)f(W_1 x(t - h(t))) + B\omega(t), \forall t > 0,$$

$$x(t) = \phi(t), t \in [-h, 0].$$

Section 3.2

3. A study on neural networks

- **Stability analysis of generalized neural networks**
- **The estimation of reachable set in Markovian jump neural networks**

3. A study on neural networks

- ❖ **Section 3.1 Stability analysis of GNNs**
 - **Problem statement**
 - **Main results**
 - **Example**

Section 3.1 : Stability analysis of GNNs

Generalized neural networks system with time-varying delays

$$\dot{x}(t) = -Ax(t) + W_0 f(W_2 x(t)) + W_1 f(W_2 x(t - h(t))), \quad \forall t > 0,$$

where $x(t) \in \mathbb{R}^n$ is the neuron state vector, $f(\cdot) \in \mathbb{R}^n = [f_1(\cdot), f_2(\cdot), \dots, f_n(\cdot)]^T$ is the activation function of the neurons. $A \in \mathbb{R}^{n \times n}$ is a positive diagonal matrix, W_0, W_1 , and $W_2 \in \mathbb{R}^{n \times n}$ are the interconnection matrices between neurons. $h(t)$ is a time-delay satisfying $0 \leq h(t) \leq h_U$ and $\dot{h}(t) < h_M$.

Used methods:

- Lyapunov method
- Linear matrix inequality(LMI) technique
- Lemma 2.1.2 (A generalized free-weighting-matrix inequality)
- Lemma 2.1.3 (Finsler's lemma), Lemma 2.1.4
- Zero equality
- Augmented zero equality approach

Objective : Stability analysis for the system (3.1.2)

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Theorem 3.1.1

For given scalars h_M and h_D , system (3.1.1) is asymptotically stable for $0 \leq h(t) \leq h_M$ and $\dot{h}(t) \leq h_D$, if there exist matrices $R, G, N \in \mathbb{S}_+^{6n}$, $Q \in \mathbb{S}_+^{3n}$, $\Lambda_i, \Delta_i \in \mathbb{D}_+^n$, $P_i \in \mathbb{S}^n$, $S_i, U_i \in \mathbb{R}^{18n \times 3n}$, $\Psi_i \in \mathbb{R}^{n \times 18n}$ ($i = 1, 2$), $J_j, L_j \in \mathbb{D}_+^n$ ($j = 1, 2, \dots, 5$) satisfying the following LMIs:

$$\begin{bmatrix} (\Gamma^\perp)^T (\Xi_{[h_M]} + \Xi_{6[h_M]}) (\Gamma^\perp) & (\Gamma^\perp)^T (h_M S_1) & (\Gamma^\perp)^T (h_M U_1) \\ \star & -h_M Q_1^{aug} & 0_{3n} \\ \star & \star & -3h_M Q_1^{aug} \end{bmatrix} < 0,$$

$$\begin{bmatrix} (\Gamma^\perp)^T (\Xi_{[0]} + \Xi_{6[0]}) (\Gamma^\perp) & (\Gamma^\perp)^T (h_M S_2) & (\Gamma^\perp)^T (h_M U_2) \\ \star & -h_M Q_2^{aug} & 0_{3n} \\ \star & \star & -3h_M Q_2^{aug} \end{bmatrix} < 0,$$

$$Q_1^{aug} \geq 0,$$

$$Q_2^{aug} \geq 0,$$

where the notations are defined as (3.1.4).

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Lyapunov-Krasovskii candidate functional:

$$V(t) = V_1 + V_2 + V_3 + V_4,$$

where

$$V_1 = \zeta_1^T(t, s) R \zeta_1(t, s),$$



$$\zeta_1(t, s) = \left[x^T(t), x^T(t - h_M), \int_{t-h_M}^t x^T(s) ds, \int_{t-h_M}^t \int_s^t x^T(u) du ds, \int_{t-h_M}^t f^T(W_2 x(s)) ds, \int_{t-h_M}^t \int_s^t f^T(W_2 x(u)) du ds \right]^T$$

$$\begin{aligned} V_2 = & \int_{t-h_M}^t \zeta_2^T(t, s) N \zeta_2(t, s) ds \\ & + 2 \sum_{i=1}^n \lambda_{1,i} \int_0^{W_{2,i} x(t)} (f_i(s) - k_i^- s) ds + \delta_{1,i} \int_0^{W_{2,i} x(t)} (k_i^+ s - f_i(s)) ds \\ & + 2 \sum_{i=1}^n \lambda_{2,i} \int_0^{W_{2,i} x(t-h_M)} (f_i(s) - k_i^- s) ds + \delta_{2,i} \int_0^{W_{2,i} x(t)} (k_i^+ s - f_i(s)) ds, \end{aligned}$$

$$\begin{aligned} V_3 = & \int_{t-h(t)}^t \zeta_3^T(t, s) G \zeta_3(t, s) ds, \\ V_4 = & \int_{t-h_M}^t \int_s^t \beta^T(u) Q \beta(u) du ds. \end{aligned}$$

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Theorem 3.1.2

For given scalars h_M and h_D , system (3.1.1) is asymptotically stable for $0 \leq h(t) \leq h_M$ and $\dot{h}(t) \leq h_D$, if there exist matrices $R \in \mathbb{S}^{6n}$, $G, N \in \mathbb{S}_+^{6n}$, $Q \in \mathbb{S}_+^{3n}$, $\Lambda_i, \Delta_i \in \mathbb{D}_+^n$, $P_i \in \mathbb{S}^n$, $S_i, U_i \in \mathbb{R}^{18n \times 3n}$, $\Psi_i \in \mathbb{R}^{n \times 18n}$ ($i = 1, 2$), $J_j, L_j \in \mathbb{D}_+^n$ ($j = 1, 2, \dots, 5$) satisfying the LMIs (3.1.5)-(3.1.8), and (3.1.33).

Proof. If $N > 0$ satisfies, the lower bound of $V(t) > \zeta_1^T(t, s)R\zeta_1(t, s) + \int_{t-h_M}^t \zeta_2^T(t, s)N\zeta_2(t, s)ds$

By Lemma 2.1.1, the lower bound of $\int_{t-h_M}^t \zeta_2^T(t, s)N\zeta_2(t, s)ds$ can be easily obtained as

$$\int_{t-h_M}^t \zeta_2^T(t, s)N\zeta_2(t, s)ds \geq \frac{1}{h_M} \left(\int_{t-h_M}^t \zeta_2(t, s)ds \right)^T N \left(\int_{t-h_M}^t \zeta_2(t, s)ds \right) \quad (3.1.33)$$

Then, the lower bound of $V(t)$ can be guaranteed to be positive.

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Theorem 3.1.3

For given scalars h_M and h_D , system (3.1.1) is asymptotically stable for $0 \leq h(t) \leq h_M$ and $\dot{h}(t) \leq h_D$, if there exist matrices $R \in \mathbb{S}^{6n}$, $G, N \in \mathbb{S}_+^{6n}$, $Q \in \mathbb{S}_+^{3n}$, $\Lambda_i, \Delta_i \in \mathbb{D}_+^n$, $P_i \in \mathbb{S}^n$, $S_i, U_i \in \mathbb{R}^{18n \times 3n}$ ($i = 1, 2$), $J_j, L_j \in \mathbb{D}_+^n$ ($j = 1, 2, \dots, 5$) satisfying the following LMIs:

$$\begin{bmatrix} \left(\Gamma_{[h_M]}^\perp\right)^T (\Xi_{[h_M]} + \Xi_{6[h_M]}) \left(\Gamma_{[h_M]}^\perp\right) & \left(\Gamma_{[h_M]}^\perp\right)^T (h_M S_1) & \left(\Gamma_{[h_M]}^\perp\right)^T (h_M U_1) \\ \star & -h_M Q_1^{aug} & 0_{3n} \\ \star & \star & -3h_M Q_1^{aug} \end{bmatrix} < 0,$$

$$\begin{bmatrix} \left(\Gamma_{[0]}^\perp\right)^T (\Xi_{[0]} + \Xi_{6[0]}) \left(\Gamma_{[0]}^\perp\right) & \left(\Gamma_{[0]}^\perp\right)^T (h_M S_2) & \left(\Gamma_{[0]}^\perp\right)^T (h_M U_2) \\ \star & -h_M Q_2^{aug} & 0_{3n} \\ \star & \star & -3h_M Q_2^{aug} \end{bmatrix} < 0,$$

where

$$\Gamma_{[h(t)]} = \begin{bmatrix} \Gamma \\ h(t)e_{10}^T - e_6^T \\ (h_M - h(t))e_{11}^T - e_7^T \end{bmatrix}.$$

The same notations are used, like Theorems 3.1.1 and 3.1.2.

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Corollary 3.1.3

For given scalars h_M and h_D , system (3.1.1) is asymptotically stable for $0 \leq h(t) \leq h_M$ and $\dot{h}(t) \leq h_D$, if there exist matrices $\hat{R} \in \mathbb{S}^{4n}$, $\hat{G}$, $\hat{N} \in \mathbb{S}_+^{4n}$, $\hat{Q} \in \mathbb{S}_+^{2n}$, $\hat{\Lambda}_i$, $\hat{\Delta}_i \in \mathbb{D}_+^n$, $\hat{P}_i \in \mathbb{S}^n$, $\hat{S}_i$, $\hat{U}_i \in \mathbb{R}^{14n \times 2n}$ ($i = 1, 2$), $\hat{J}_j \in \mathbb{D}_+^n$, $\hat{L}_j \in \mathbb{D}_+^n$ ($j = 1, 2, \dots, 5$) satisfying the following LMIs:

$$\begin{bmatrix} \left(\hat{\Gamma}_{[h_M]}^\perp\right)^T \hat{\Xi}_{[h_M]} \left(\hat{\Gamma}_{[h_M]}^\perp\right) & \left(\hat{\Gamma}_{[h_M]}^\perp\right)^T \left(h_M \hat{S}_1\right) & \left(\hat{\Gamma}_{[h_M]}^\perp\right)^T \left(h_M \hat{U}_1\right) \\ \star & -h_M \hat{Q}_1^{aug} & 0_{2n} \\ \star & \star & -3h_M \hat{Q}_1^{aug} \end{bmatrix} < 0,$$

$$\begin{bmatrix} \left(\hat{\Gamma}_{[0]}^\perp\right)^T \hat{\Xi}_{[0]} \left(\hat{\Gamma}_{[0]}^\perp\right) & \left(\hat{\Gamma}_{[0]}^\perp\right)^T \left(h_M \hat{S}_2\right) & \left(\hat{\Gamma}_{[0]}^\perp\right)^T \left(h_M \hat{U}_2\right) \\ \star & -h_M \hat{Q}_2^{aug} & 0_{2n} \\ \star & \star & -3h_M \hat{Q}_2^{aug} \end{bmatrix} < 0,$$

$$\hat{Q}_1^{aug} \geq 0,$$

$$\hat{Q}_2^{aug} \geq 0,$$

$$\hat{R} + \frac{1}{h_M} \left[\hat{e}_3, \hat{e}_1 - \hat{e}_2, \hat{e}_4, h_M \hat{e}_1 - \hat{e}_3 \right] \hat{N}$$

$$\left[\hat{e}_3, \hat{e}_1 - \hat{e}_2, \hat{e}_4, h_M \hat{e}_1 - \hat{e}_3 \right]^T > 0,$$

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Corollary 3.1.3 (Cont.)

For given scalars h_M and h_D , system (3.1.1) is asymptotically stable for $0 \leq h(t) \leq h_M$ and $\dot{h}(t) \leq h_D$, if there exist matrices $\hat{R} \in \mathbb{S}^{4n}$, $\hat{G}$, $\hat{N} \in \mathbb{S}_+^{4n}$, $\hat{Q} \in \mathbb{S}_+^{2n}$, $\hat{\Lambda}_i$, $\hat{\Delta}_i \in \mathbb{D}_+^n$, $\hat{P}_i \in \mathbb{S}^n$, $\hat{S}_i$, $\hat{U}_i \in \mathbb{R}^{14n \times 2n}$ ($i = 1, 2$), $\hat{J}_j \in \mathbb{D}_+^n$, $\hat{L}_j \in \mathbb{D}_+^n$ ($j = 1, 2, \dots, 5$) satisfying the following LMIs:

where

$$\hat{\Gamma}_{[h(t)]} = \begin{bmatrix} \hat{\Gamma} \\ h(t)\hat{e}_{10}^T - \hat{e}_6^T \\ (h_M - h(t))\hat{e}_{11}^T - \hat{e}_7^T \end{bmatrix},$$

where the notations are defined as (3.1.40).

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Lyapunov-Krasovskii candidate functional:

$$\hat{V}(t) = \hat{V}_1 + \hat{V}_2 + \hat{V}_3 + \hat{V}_4,$$

where

$$\hat{V}_1 = \hat{\zeta}_1^T(t, s) \hat{R} \hat{\zeta}_1(t, s),$$



$$\hat{\zeta}_1(t, s) = \left[x^T(t), x^T(t - h_M), \int_{t-h_M}^t x^T(s) ds, \int_{t-h_M}^t \int_s^t x^T(u) du ds \right]^T,$$

$$\begin{aligned} \hat{V}_2 = & \int_{t-h_M}^t \hat{\zeta}_2^T(t, s) \hat{N} \hat{\zeta}_2(t, s) ds \\ & + 2 \sum_{i=1}^n \left(\hat{\lambda}_{1,i} \int_0^{W_{2,i}x(t)} (f_i(s) - k_i^- s) ds + \hat{\delta}_{1,i} \int_0^{W_{2,i}x(t)} (k_i^+ s - f_i(s)) ds \right) \\ & + 2 \sum_{i=1}^n \left(\hat{\lambda}_{2,i} \int_0^{W_{2,i}x(t-h_M)} (f_i(s) - k_i^- s) ds + \hat{\delta}_{2,i} \int_0^{W_{2,i}x(t-h_M)} (k_i^+ s - f_i(s)) ds \right), \end{aligned}$$

$$\hat{V}_3 = \int_{t-h(t)}^t \hat{\zeta}_3^T(t, s) \hat{G} \hat{\zeta}_3(t, s) ds,$$

$$\hat{V}_4 = \int_{t-h_M}^t \int_s^t \hat{\beta}^T(u) \hat{Q} \hat{\beta}(u) du ds.$$

3. A study on neural networks

- ❖ **Section 3.1 Stability analysis of GNNs**
 - Problem statement
 - **Main results**
 - Example

Th 3.1.1	Constructing LKFs based the activation function	$\zeta_1(t, s) = \left[\dots, \int_{t-h_M}^t f^T(W_2 x(s)) ds, \int_{t-h_M}^t \int_s^t f^T(W_2 x(u)) duds \right]^T$
Th 3.1.2	Positiveness	$V(t) > \zeta_1^T(t, s) R \zeta_1(t, s) + \int_{t-h_M}^t \zeta_2^T(t, s) N \zeta_2(t, s) ds$
Th 3.1.3	Augmented zero equality approach	$\Gamma_{[h(t)]} = \begin{bmatrix} \Gamma \\ h(t)e_{10}^T - e_6^T \\ (h_M - h(t))e_{11}^T - e_7^T \end{bmatrix}$
Cor 3.1.3	No the activation function terms	

3. A study on neural networks

- ❖ **Section 3.1 Stability analysis of GNNs**
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Example 3.1.2 Let us consider the system (3.1.1) :

$$\dot{x}(t) = -Ax(t) + W_0 f(W_2 x(t)) + W_1 f(W_2 x(t - h(t))), \quad \forall t \geq 0 \quad (3.1.1)$$

$$A = \begin{bmatrix} 1.5 & 0 \\ 0 & 0.7 \end{bmatrix}, \quad W_0 = \begin{bmatrix} 0.0503 & 0.0454 \\ 0.0987 & 0.2075 \end{bmatrix}, \quad W_1 = \begin{bmatrix} 0.2381 & 0.9320 \\ 0.0388 & 0.5062 \end{bmatrix},$$

$$W_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad K_p = \text{diag}\{0.3, 0.8\}, \quad K_m = 0_2.$$

(I) A comparison of maximum delay bounds in generalized neural networks system

[Theorem 3.1.1 – Theorem 3.1.3]

(II) A comparison of results according to augmented LKFs, including the activation function terms

[Theorem 3.1.3, Corollary 3.1.3]

3. A study on neural networks

- ❖ **Section 3.1 Stability analysis of GNNs**
 - Problem statement
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Example 3.1.2 (Cont.)

- (I) A comparison of maximum delay bounds in generalized neural networks system [Theorem 3.1.1 – Theorem 3.1.3]

Table 3.1.2 Maximum bounds of NNTVDs with different h_D (Example 3.1.2)

h_D	0.40	0.45	0.50	0.55
M.J Park <i>et al.</i> [11]	17.2697	12.0698	10.2903	9.3879
Z. Feng <i>et al.</i> [24]	18.8262	12.8039	10.6230	9.5038
B. Yang <i>et al.</i> [26]	20.8249	14.0082	11.5834	10.2690
J.A Wang <i>et al.</i> [25]	21.0344	14.4707	11.9667	10.6439
O.M Kwon <i>et al.</i> [12]	27.7002	19.5050	16.4611	14.9900
Ours (Theorem 3.1.1)	17.2194	12.2222	10.5582	9.7551
Ours (Theorem 3.1.2)	17.5781	12.5651	10.8613	10.0089
Ours (Theorem 3.1.3)	32.3833	22.5316	19.5912	18.3621

3. A study on neural networks

- ❖ **Section 3.1 Stability analysis of GNNs**
 - Problem statement
 - Main results
 - **Example**

Example 3.1.2 (Cont.)

(II) A comparison of results according to augmented LKFs, including the activation function terms
[Theorem 3.1.3, Corollary 3.1.3]

Table 3.1.2 Maximum bounds of NNTVDs with different h_D (Example 3.1.2)

h_D	0.40	0.45	0.50	0.55
Ours (Theorem 3.1.3)	32.3833	22.5316	19.5912	18.3621
Ours (Corollary 3.1.3)	31.4262	21.2865	17.7803	16.1689

3. A study on neural networks

- ❖ Section 3.1 Stability analysis of GNNs
 - Problem statement
 - Main results
 - Example

Example 3.1.2 (Cont.)

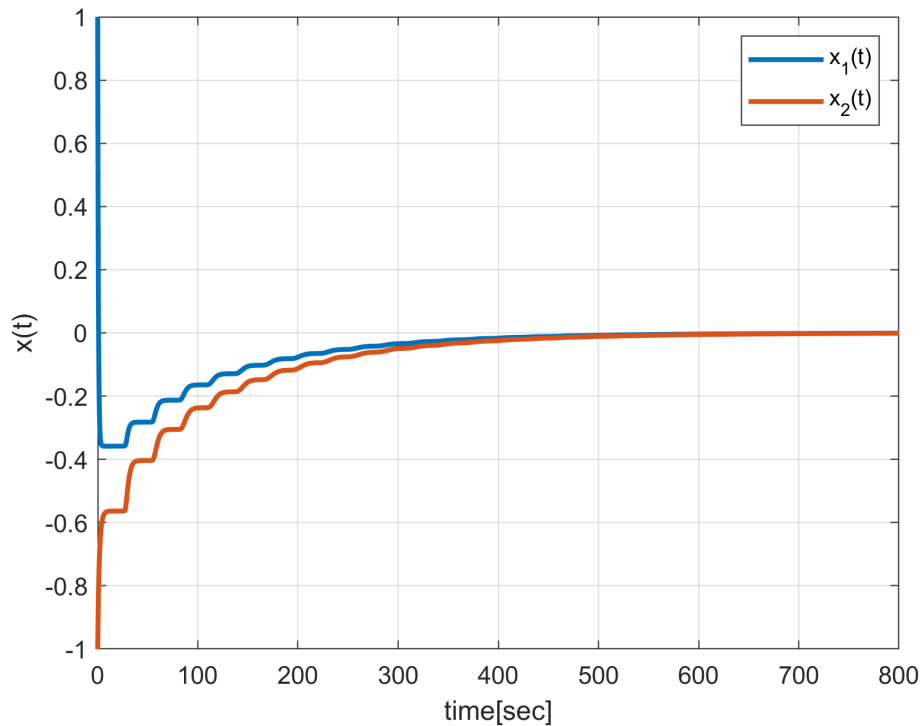


Figure 3.1.1 : State trajectories of system (3.1.1) with $h(t) = 0.3\sin^2(t) + 32.0833$

3. A study on neural networks

- ❖ Section 3.2 The estimation of the reachable set
 - Problem statement
 - Main results
 - Example

Section 3.2 : The estimation of the reachable set

Markovian jump neural networks with time-varying delays

$$\begin{aligned}\dot{x}(t) &= -A(r_t)x(t) + W_0(r_t)f(W_1x(t)) + W_2(r_t)f(W_1x(t-h(t))) + B\omega(t), \quad \forall t > 0, \\ x(t) &= \phi(t), t \in [-h, 0],\end{aligned}$$

where $x(t) \in \mathbb{R}^n$ is the neuron state vector, the initial condition $\phi(t)$ is a continuously differentiable function. The activation function is defined as $f(\cdot) \in \mathbb{R}^n = [f_1(\cdot), f_2(\cdot), \dots, f_n(\cdot)]^T$. $h(t)$ means the time-varying delays satisfying $0 \leq h(t) \leq h_U$ and $\dot{h}(t) < h_M$. $\omega^T(t) \in \mathbb{R}^l$ means external input with the form of bounded peak $\omega^T(t)\omega(t) \leq \omega_M$ where ω_M is a constant. $A(r_t) \in \mathbb{R}^{n \times n}$ is a positive diagonal matrix, $W_0(r_t)$, W_1 , and $W_2(r_t) \in \mathbb{R}^{n \times n}$ are the interconnection matrices between neurons.

3. A study on neural networks

- ❖ **Section 3.2 The estimation of the reachable set**
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 - **Example**

r_t is a right-continuous Markov chain on the probability space having the values in a finite-state space $N = \{1, 2, \dots, m\}$ with generator $\Pi = \{\pi_{pq}\}$, which is represented by

$$P\{r_{t+\Delta t} = q | r_t = p\} = \begin{cases} \pi_{pq}\Delta t + o(\Delta t), & p \neq q \\ 1 + \pi_{pp}\Delta t + o(\Delta t), & p = q \end{cases}$$

where $\Delta t > 0$ and $\lim_{\Delta t \rightarrow 0} \frac{o(\Delta t)}{\Delta t} = 0$. If $p \neq q$, $\pi_{pq} \geq 0$ is the transition rate from mode p at time t to mode q at time $t + \Delta t$, and $\pi_{pp} = -\sum_{q=1, q \neq p}^m \pi_{pq}$.

Used methods:

- Lyapunov method
- Linear matrix inequality(LMI) technique
- Lemma 2.1.2 (A generalized free-weighting-matrix inequality)
- Lemma 2.1.3 (Finsler's lemma)
- Lemma 2.1.4
- Zero equality
- Augmented zero equality approach

Objective : The estimation of reachable set for the system (3.2.1)

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Remark 4 In [46], a simple approximation to minimize the volume ϵ of (3.2.7) was suggested as $\delta I \leq P$ and this is equivalent as follows:

$$\begin{aligned} & \text{minimize } \bar{\delta} = \frac{1}{\delta}, \\ & \text{subject to } \begin{bmatrix} \bar{\delta} I & I \\ I & P \end{bmatrix} > 0, \text{ and LMIs (3.2.8) and (3.2.9).} \end{aligned}$$

Likewise [47], the optimization toolbox is used to solve the optimization value of α by `fminsearch.m`. The obtained results can be used to confirm the ellipsoid in the following numerical examples.

[46] E. Fridman, U. Shaked, On reachable sets for linear systems with delay and bounded peak inputs, *Automatica* **39** (2003) 2005-2010.

[47] J.H. Kim, Improved ellipsoidal bound of reachable sets for time-delayed linear systems with disturbances, *Automatica* **44** (2008) 2940-2943.

3. A study on neural networks

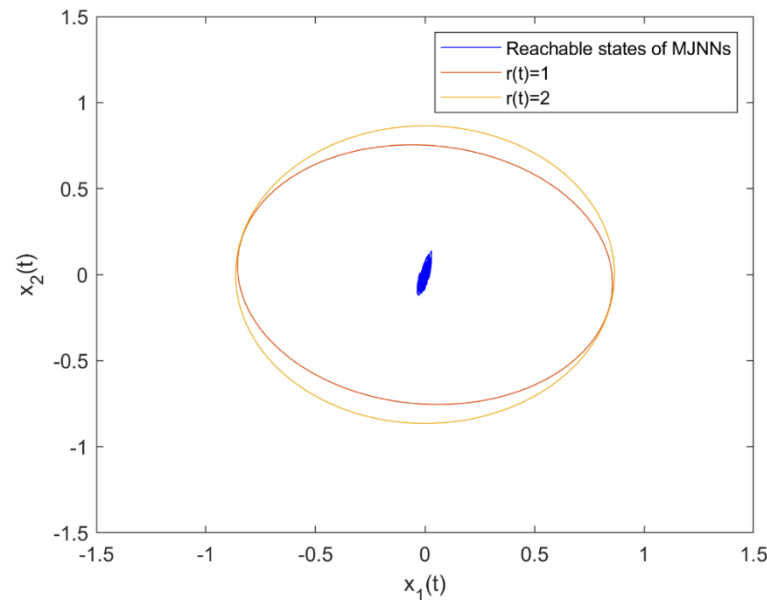
- ❖ Section 3.2 The estimation of the reachable set
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Definition 1. For $\forall t \geq 0$, the reachable set of system (3.2.1) is denoted as follows:

$$R_x = \{x(t) | x(t) \text{ and } \omega(t) \text{ satisfy (1) with (2) and (3)}\} \quad (1)$$

and an ellipsoid used to bound the reachable set of Markovian jump NNTVDs (3.2.1) is given in the form of

$$\epsilon(P) = \{\xi | \xi^T P \xi \leq 1, \xi \in \mathbb{R}^n, P > 0\}.$$



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- ❖ Section 3.2 The estimation of the reachable set
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Theorem 3.2.1

For given scalars h_M and μ , system (3.2.1) is asymptotically stable for $0 \leq h(t) \leq h_M$ and $\dot{h}(t) \leq \mu$, if there exist scalar $\alpha > 0$, matrices $U_m \in \mathbb{S}_+^n$, $P, R, Q_1 \in \mathbb{S}_+^{3n}$, $Q_2 \in \mathbb{S}_+^{2n}$, $\Lambda_i, \Delta_i \in \mathbb{D}_+^n$, $P_i \in \mathbb{S}^n$, $S_i, N_i \in \mathbb{R}^{(19n-l) \times 3n}$, $\Psi_i \in \mathbb{R}^{n \times (19n-l)}$ ($i = 1, 2$), $H_j, F_j \in \mathbb{D}_+^n$ ($j = 1, 2, \dots, 6$) satisfying the following LMIs:

$$\begin{bmatrix} (\Gamma^\perp)^T \left(\Xi_{[h_M]}^m - \beta e_{19} e_{19}^T \right) (\Gamma^\perp) & (\Gamma^\perp)^T (h_M S_1) & (\Gamma^\perp)^T (h_M N_1) \\ \star & -h_M e^{\alpha h_M} R_{aug1} & 0_{3n} \\ \star & \star & -3h_M e^{\alpha h_M} R_{aug1} \end{bmatrix} < 0,$$

$$\begin{bmatrix} (\Gamma^\perp)^T \left(\Xi_{[0]}^m - \beta e_{19} e_{19}^T \right) (\Gamma^\perp) & (\Gamma^\perp)^T (h_M S_2) & (\Gamma^\perp)^T (h_M N_2) \\ \star & -h_M e^{\alpha h_M} R_{aug2} & 0_{3n} \\ \star & \star & -3h_M e^{\alpha h_M} R_{aug2} \end{bmatrix} < 0,$$

$$R_{aug1} \geq 0,$$

$$R_{aug2} \geq 0,$$

where the notations are defined as (3.2.7).

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- ❖ Section 3.2 The estimation of the reachable set
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The notations of Theorems 3.2.1 - 3.2.2 are as follows

$$\begin{aligned}
 \zeta(t) = & \left[x^T(t), x^T(t-h(t)), x^T(t-h_M), \dot{x}^T(t), \dot{x}^T(t-h_M), \frac{1}{h(t)} \int_{t-h(t)}^t x^T(s) ds, \right. \\
 & \frac{1}{h_M-h(t)} \int_{t-h_M}^{t-h(t)} x^T(s) ds, f^T(W_1 x(t)), f^T(W_1 x(t-h(t))), \\
 & f^T(W_1 x(t-h_M)), \frac{1}{h(t)} \int_{t-h(t)}^t \int_s^t x^T(u) du ds, \\
 & \frac{1}{h_M-h(t)} \int_{t-h_M}^{t-h(t)} \int_s^{t-h(t)} x^T(u) du ds, \frac{1}{h(t)} \int_{t-h(t)}^t \int_s^t f^T(W_1 x(u)) du ds, \\
 & \frac{1}{h_M-h(t)} \int_{t-h_M}^{t-h(t)} \int_s^{t-h(t)} f^T(W_1 x(u)) du ds, \\
 & \left. \int_{t-h(t)}^t x^T(s) ds, \int_{t-h_M}^{t-h(t)} x^T(s) ds, \omega^T(t) \right]^T, \\
 \eta_1(t) = & \left[x^T(t), x^T(t-h_M), \int_{t-h_M}^t x^T(s) ds \right]^T, \\
 \eta_2(s) = & \left[x^T(s), \dot{x}^T(s), f^T(W_1 x(s)) \right]^T,
 \end{aligned}$$

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Lyapunov-Krasovskii candidate functionals:

$$V(x_t, m) = V_1(x_t, m) + \sum_{i=2}^4 V_i(x_t),$$

where

$$V_1(x_t, m) = x^T(t)U_mx(t) + \eta_1^T(t)P\eta_1(t),$$

$$\begin{aligned} V_2(x_t) = & \int_{t-h_M}^t e^{\alpha(s-t)} \eta_2^T(s) Q_1 \eta_2(s) ds \\ & + 2e^{-\alpha t} \sum_{i=1}^n \int_0^{W_{1,i}x(t)} [\lambda_{1,i} (f_i(s) - k_i^- s) + \delta_{1,i} (k_i^+ s - f_i(s))] ds \\ & + 2e^{-\alpha t} \sum_{i=1}^n \int_0^{W_{1,i}x(t-h_M)} [\lambda_{2,i} (f_i(s) - k_i^- s) + \delta_{2,i} (k_i^+ s - f_i(s))] ds, \end{aligned}$$

$$\begin{aligned} V_3(x_t) = & \int_{t-h(t)}^t \begin{bmatrix} x(s) \\ f(W_1 x(s)) \end{bmatrix}^T Q_2 \begin{bmatrix} x(s) \\ f(W_1 x(s)) \end{bmatrix} ds, \\ V_4(x_t) = & \int_{t-h_M}^t \int_s^t e^{\alpha(u-t)} \eta_2^T(u) R \eta_2(u) du ds. \end{aligned}$$

2. Preliminaries

- Lyapunov-Krasovskii's theorem
- Neural networks system
- Markovian jump process
- **Fact and lemmas**
- Preview of the systems

Lemma 2.1.5 [47]

Let $V(x(t))$ be a differentiable Lyapunov function for system with $V(x(0))=0$ and external inputs $\omega(t)$ satisfying $\omega^T(t)\omega(t) \leq \omega_M^2$. If there exists scalars $\alpha > 0$ and $\beta > 0$ such that

$$\dot{V}(x(t)) + \alpha V(x(t)) - \beta \omega^T(t)\omega(t) \leq 0, \text{ then } V(x(t)) \leq \frac{\beta}{\alpha} \omega_M^2.$$

where

$$V_1(x_t, m) = x^T(t)U_m x(t) + \eta_1^T(t)P\eta_1(t),$$

$$\begin{aligned} V_2(x_t) &= \int_{t-h_M}^t e^{\alpha(s-t)} \eta_2^T(s) Q_1 \eta_2(s) ds \\ &\quad + 2e^{-\alpha t} \sum_{i=1}^n \int_0^{W_{1,i}x(t)} [\lambda_{1,i} (f_i(s) - k_i^- s) + \delta_{1,i} (k_i^+ s - f_i(s))] ds \\ &\quad + 2e^{-\alpha t} \sum_{i=1}^n \int_0^{W_{1,i}x(t-h_M)} [\lambda_{2,i} (f_i(s) - k_i^- s) + \delta_{2,i} (k_i^+ s - f_i(s))] ds, \end{aligned}$$

$$V_3(x_t) = \int_{t-h(t)}^t \begin{bmatrix} x(s) \\ f(W_1 x(s)) \end{bmatrix}^T Q_2 \begin{bmatrix} x(s) \\ f(W_1 x(s)) \end{bmatrix} ds,$$

$$V_4(x_t) = \int_{t-h_M}^t \int_s^t e^{\alpha(u-t)} \eta_2^T(u) R \eta_2(u) du ds.$$

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Theorem 3.2.2

For given scalars h_M and μ , system (3.2.1) is asymptotically stable for $0 \leq h(t) \leq h_M$ and $\dot{h}(t) \leq \mu$, if there exist scalar $\alpha > 0$, matrices $U_m \in \mathbb{S}_+^n$, $P, R, Q_1 \in \mathbb{S}_+^{3n}$, $Q_2 \in \mathbb{S}_+^{2n}$, $\Lambda_i, \Delta_i \in \mathbb{D}_+^n$, $P_i \in \mathbb{S}^n$, $S_i, N_i \in \mathbb{R}^{(19n-l) \times 3n}$, $H_j, F_j \in \mathbb{D}_+^n$ ($j = 1, 2, \dots, 6$) satisfying (3.2.10), (3.2.11), and the following LMIs:

$$\begin{bmatrix} \left(\Gamma_{m[h]}^\perp\right)^T \left(\Xi_{[h_M]}^m - \beta e_{19} e_{19}^T\right) \left(\Gamma_{m[h]}^\perp\right) & \left(\Gamma_{m[h]}^\perp\right)^T (h_M S_1) & \left(\Gamma_{m[h]}^\perp\right)^T (h_M N_1) \\ \star & -h_M e^{\alpha h_M} R_{aug1} & 0_{3n} \\ \star & \star & -3h_M e^{\alpha h_M} R_{aug1} \end{bmatrix} < 0,$$

$$\begin{bmatrix} \left(\Gamma_{m[0]}^\perp\right)^T \left(\Xi_{[0]}^m - \beta e_{19} e_{19}^T\right) \left(\Gamma_{m[0]}^\perp\right) & \left(\Gamma_{m[0]}^\perp\right)^T (h_M S_2) & \left(\Gamma_{m[0]}^\perp\right)^T (h_M N_2) \\ \star & -h_M e^{\alpha h_M} R_{aug2} & 0_{3n} \\ \star & \star & -3h_M e^{\alpha h_M} R_{aug2} \end{bmatrix} < 0,$$

subject to $\Gamma_{m[h(t)]} = \begin{bmatrix} \Gamma \\ h(t) e_6^T - e_{17}^T \\ (h_M - h(t)) e_7^T - e_{18}^T \end{bmatrix}.$

The same notations are used, like Theorem 3.2.1.

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- ❖ **Section 3.2 The estimation of the reachable set**
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Example 3.2.1 Let us consider the system (3.2.1):

$$\dot{x}(t) = -A(r_t)x(t) + W_0(r_t)f(W_1x(t)) + W_2(r_t)f(W_1x(t - h(t))) + B\omega(t), \quad \forall t > 0 \quad (3.2.1)$$

$$\begin{aligned} A_1 &= \begin{bmatrix} 2 & 0 \\ 0 & 3.5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2.2 & 0 \\ 0 & 1.8 \end{bmatrix}, \quad B_1 = B_2 = \begin{bmatrix} 0.0403 \\ 0.6771 \end{bmatrix}, \\ W_{01} &= \begin{bmatrix} -1 & 0.5 \\ 0.5 & -1 \end{bmatrix}, \quad W_{02} = \begin{bmatrix} -1 & 1 \\ 0.5 & -1 \end{bmatrix}, \quad W_1 = I_2, \\ W_{21} &= \begin{bmatrix} -0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}, \quad W_{22} = \begin{bmatrix} -0.5 & 0.6 \\ 0.7 & 0.8 \end{bmatrix}, \\ K_p &= \text{diag}\{0.4, 0.8\}, \quad K_m = 0_2. \end{aligned}$$

The transition probabilities matrix is given by $\Pi = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix} = \begin{bmatrix} -0.2 & 0.2 \\ 0.8 & -0.8 \end{bmatrix}$.

The external input of MJNNs (3.2.1) is assumed to satisfy (3.2.3) with $\omega_M = 1$.

(I) A comparison of allowable smaller bounds in Markovian jump neural networks system [Theorem 3.2.1 – Theorem 3.2.2]

(II) The figures about the ellipsoid areas of the reachable set states (Example 3.2.1)

3. A study on neural networks

- ❖ Section 3.2 The estimation of the reachable set
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Example 3.2.1 (Cont.)

(I) A comparison of allowable smaller bounds in Markovian jump neural networks system [Theorem 3.2.1 – Theorem 3.2.2]

Table 1: Allowable smaller bounds $\bar{\delta}$ for various μ and $h_M = 2$ (Example 3.2.1).

μ	0.2	0.4	0.6	0.8
W.J. Lin <i>et al.</i> [44]	0.6821 ($\alpha = 0.44$)	0.8578 ($\alpha = 0.39$)	1.2302 ($\alpha = 0.31$)	2.2204 ($\alpha = 0.22$)
Ours (Theorem 3.2.1)	0.5158 ($\alpha = 0.4124$)	0.5965 ($\alpha = 0.3817$)	0.7221 ($\alpha = 0.3526$)	0.8412 ($\alpha = 0.3252$)
Ours (Theorem 3.2.2)	0.4698 ($\alpha = 0.4462$)	0.5031 ($\alpha = 0.4550$)	0.5301 ($\alpha = 0.4624$)	0.5469 ($\alpha = 0.4594$)

32% ↓

42% ↓

57% ↓

75% ↓

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- ❖ Section 3.2 The estimation of the reachable set
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Example 3.2.1 (Cont.)

(I) A comparison of allowable smaller bounds in Markovian jump neural networks system [Theorem 3.2.1 – Theorem 3.2.2]

Table 1: Allowable smaller bounds $\bar{\delta}$ for various μ and $h_M = 3$ (Example 3.2.1).

μ	0.2	0.4	0.6	0.8
W.J. Lin <i>et al.</i> [44]	0.9443 ($\alpha = 0.30$)	1.2456 ($\alpha = 0.26$)	1.9351 ($\alpha = 0.21$)	5.0915 ($\alpha = 0.12$)
Ours (Theorem 3.2.1)	0.7480 ($\alpha = 0.2807$)	0.9034 ($\alpha = 0.2516$)	1.2175 ($\alpha = 0.2156$)	1.9624 ($\alpha = 0.1698$)
Ours (Theorem 3.2.2)	0.7018 ($\alpha = 0.2998$)	0.7939 ($\alpha = 0.2875$)	0.9274 ($\alpha = 0.2702$)	1.1454 ($\alpha = 0.2437$)

26% ↓

36% ↓

52% ↓

78% ↓

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- ❖ Section 3.2 The estimation of the reachable set
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Example 3.2.1 (Cont.)

(II) The figures about the ellipsoid areas of the reachable set states (Example 3.2.1)

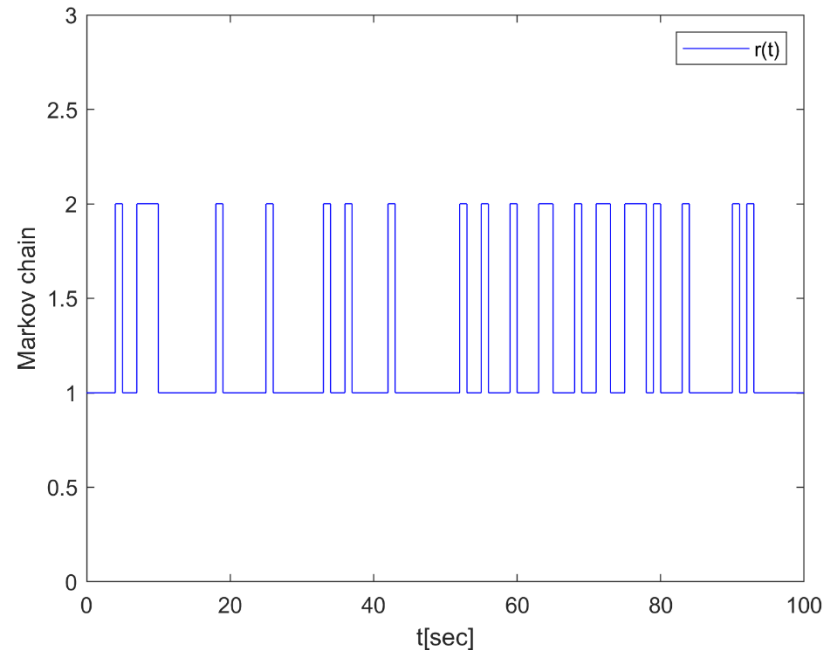


Figure 3.2.1 : Random jumping mode $r(t)$ of system (3.2.1) (Example 3.2.1)

3. A study on neural networks

- ❖ **Section 3.2 The estimation of the reachable set**
 - Problem statement
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Example 3.2.1 (Cont.)

(II) The figures about the ellipsoid areas of the reachable set states (Example 3.2.1)

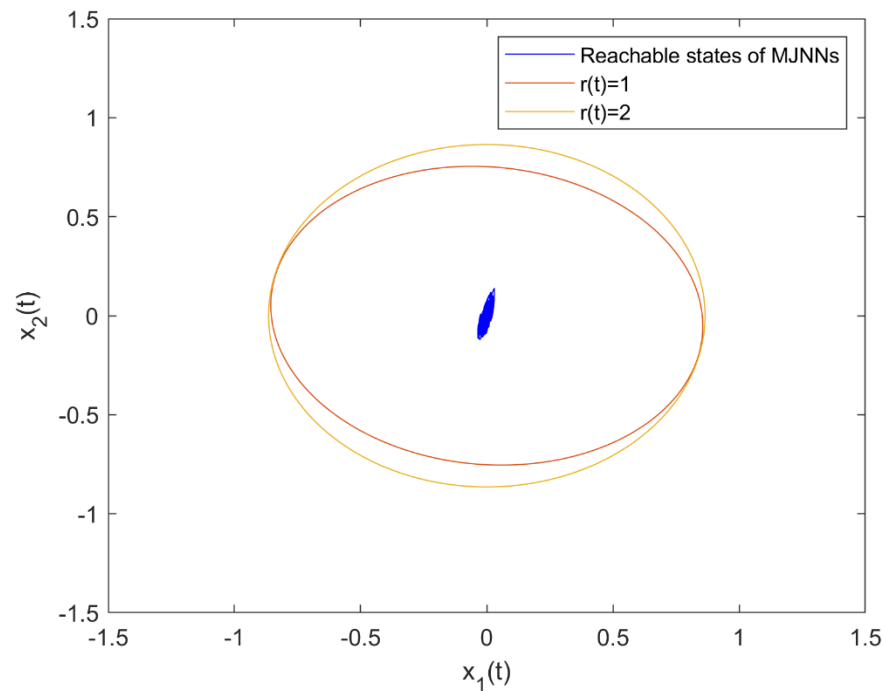


Figure 3.2.2 : Reachable states of system (3.2.1) (Example 3.2.1)

4. Conclusion

4. Conclusions

Stability analysis and the estimation of the reachable set for delayed neural networks systems based on Lyapunov methods, theorems and corollaries in terms of LMIs.

- A newly augmented Lyapunov-Krasovskii functional candidates are presented.
- Following systems are utilized.
 - Generalized neural networks system
 - Time-varying delay
 - Stability analysis
 - Markovian jump neural networks system
 - Time-varying delay
 - The estimation of the reachable set
 - Markovian jump process
- Numerical examples are included to show the effectiveness of proposed methods.

Thank you for your attention.