

공학석사학위논문

일반화된 신경망 시스템에 대한
안정성 분석과 도달 가능한 집합 추정에 관한 연구

A Study on Stability Analysis and
The Estimation of The Reachable Set
for Generalized Neural Networks System

충 북 대 학 교 대 학 원

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A Study on Stability Analysis and The Estimation of The Reachable Set for Generalized Neural Networks Systems*

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Abstract

In this thesis, the problem of stability analysis is dealt with in a generalized neural networks system, and the estimation of the reachable set is applied in Markovian jump neural networks systems with time-varying delays. The stability analysis is suggested through the linear matrix inequalities. Based on a concept of stability analysis, the optimized solution of the reachable set is proposed. By constructing improved augmented Lyapunov-Krasovskii functionals and utilizing well-known lemmas to verify maximum delay bounds and minimum reachable sets in neural networks systems, the proposed method results are derived. The numerical examples show the effectiveness and superiority of the proposed methods.

Keywords: Neural networks systems, Reachable set, Time-varying delay, Stability analysis, Lyapunov-Krasovskii functionals

*A thesis for the degree of master in August 2022.

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Notations

$\mathbb{R}^n$	The Euclidean space
$\mathbb{S}^n$	Symmetric $n \times n$ matrices
$\mathbb{D}^n$	Diagonal $n \times n$ matrices
$\mathbb{R}_+$	Positive definite matrix
$\mathbb{R}^{m \times n}$	Real $m \times n$ matrices
$X > 0$ ($X < 0$)	X is positive definite (negative definite)
I_n	$n \times n$ identity matrix
0_n	$n \times n$ zero matrix
$0_{n \times m}$	$n \times m$ zero matrix
$X^\perp \in \mathbb{R}^{n \times (n-r)}$	Right orthogonal complement of $X \in \mathbb{R}^{m \times n}$, such that $\text{rank}(X) = r$; <i>i.e.</i> , $XX^\perp = 0$
X^T	Transpose of matrix X
X^{-1}	Inverse of matrix X
$X_{[f(t)]}$	$X_{[f(t)]}$ is affinely dependent for $f(t)$, $X_{[f(t)]} = X_1 + f(t)X_2$
$\text{diag}\{\dots\}$	Block diagonal matrix
$\text{Sym}(X)$	Sum of X and its symmetric matrix X^T
$\star$	Symmetric terms in symmetric matrices or quadratic forms
■	End of proof
LMI	Linear Matrix Inequalities
LKF	Lyapunov-Krasovskii functionals
NNTVDs	Neural networks with time-varying delays

I. Introduction

1.1 Motivation

For recent two decades, neural networks have attracted many attentions for stability analysis with time-varying delays in science and engineering fields such as pattern recognition, secure communication, image processing, associative memories, etc [4]. It is well known that the development of NNTVDs influences dynamic systems. Therefore, the study of neural networks is an influential part that can be used in various fields. For detail, many researchers illustrate the importance of neural networks [5],[6].

Time-delay frequently occurs due to the finite switching speed of amplifiers and the communication among neurons in neural networks. It can give poor performances or instability such as oscillations or destruction for a system [7]. Therefore, there is a need to solve these problems, and the study on NNTVDs has occupied one of the critical areas. Recently, several researches have been actively strived to derive the enhanced delay-dependent stability conditions for a time-delay system since they can lead to less conservative results than the delay-independent ones [5], [8], [9].

In the time-delay system, it is usually practical to obtain stability conditions that utilize Lyapunov stability theory. This crucial method uses tractable LMIs. LMIs technique helps the stability condition to check easily by solving a certain convex optimization problem. To get the improved effect, the following three items are typically required in Lyapunov theory. One is constructing the proper LKFs. Another is devising an integral inequality that provides lower bounds for integral terms. The other is utilizing some mathematical techniques to improve the stability results.

Firstly, it is well known that constructing the proper LKFs is a critical and very important job to enlarge the maximum delay bounds. Significantly, augmented LKFs can derive less conservative results by providing various cross-terms of the augmented vectors. Through the newly proposed augmented LKFs, including time-varying delay, the advantages of enhanced stability results for NNTVDs have been represented in [10]. Until recently, the augmented LKFs usually have been used in a variety of systems such as neural networks [11], [12], T-S fuzzy systems [13], sampled-data control systems [14], and so on.

Secondly, the integral inequality converts the quadratic form to LMIs and more rigorously estimates the derivative of the LKF for yielding less conservative results. Before two decades, Jensen's inequality has utilized a tighter estimating method in the works [15],[16],[17]. After that, Wirtinger-based inequality(WBI), which shows improved results than Jensen's inequality, was proposed in [18]. This technique has been widely used in various fields, and the stability analysis for NNTVDs has also been used [12],[19],[20]. After the WBI, many recent researches, such as generalized integral inequality, auxiliary function-based inequality, free-matrix-based inequality, and so on, have been introduced in [21],[22],[23],[24],[25]. Very recently, the generalized free-weighting-matrix inequality has been a main key to expand less conservative regions for NNTVDs [11],[26].

Finally, the following mathematical techniques derive more influential stability conditions. A zero equality [27] has been extensively utilized in many works [16], [28], [29]. The reciprocally convex approach is another mathematical technique that gets noticeable attention [30]. This technique can get a lower bound of integral terms and be critical to derive less conservative conditions. Since then, several improved methods have been introduced in NNTVDs [28],[31],[32]. The generalized zero equality approach [33] utilizes the relationship between the integral terms and their integration by parts. Using zero equality can lead to the improved results [11] and [34]. But, this technique has a disadvantage: it has many free variables that are proportional

to the size of the augmented vector. Very recently, the augmented zero equality approach(AZEA) was proposed to eliminate free variables of the zero equalities and to obtain less conservative results [35]. Previous studies on NNTVDs have many free variables since they used zero equalities created using the relationship of augmented vectors. By utilizing the AZEA, the number of free variables is reduced and less conservative results are obtained.

With motivations mentioned in above discussions, this paper investigated the problem of stability analysis in generalized neural networks and the estimation of the reachable set in Markovian jump neural networks. In Theorem 3.1.1-3.1.3, a stability analysis method with better results is suggested based on previous studies in NNTVDs. By constructing expended augmented LKFs and utilizing some lemmas, the LMIs conditions are derived. Comparison with the results of Corollary 3.1.3 also demonstrates the superiority and novelty of the LKFs proposed in section 3.1. Based on the principles of section 3.1, the estimation method of the reachable set is proposed in section 3.2. It aims to minimize the area of the reachable set while optimizing specific parameter values in Markovian jump NNVTDs. Via various numerical examples, the superiority, effectiveness and necessity of the proposed methods are explained.

Table 1.1.1 shows the considered systems in this thesis.

Table 1.1.1: Considered systems in this thesis.

Chapter	Systems	page
3.1	$\dot{x}(t) = -Ax(t) + W_0f(W_2x(t)) + W_1f(W_2x(t - h(t)))$	11
3.2	$\dot{x}(t) = -A(r_t)x(t) + W_0(r_t)f(W_1x(t))$ $+ W_2(r_t)f(W_1x(t - h(t))) + B(r_t)\omega(t)$	33

II. Preliminaries

In this section, useful lemmas are introduced to understand the proof of the main results. The following basic concepts need to explain each neural networks systems in this thesis. Based on Lyapunov-Krasovskii's theorem, they can be applied to given systems.

2.1 Mathematical Preliminaries

Let us introduce a well-known fact and the lemmas for proof.

Fact 2.1.1 (Schur Complement) [1] *Given constant matrices $\Sigma_1, \Sigma_2, \Sigma_3$ with $\Sigma_1 = \Sigma_1^T < 0$ and $\Sigma_2 = \Sigma_2^T > 0$, then $\Sigma_1 + \Sigma_3^T \Sigma_2^{-1} \Sigma_3 < 0$ if and only if*

$$\begin{bmatrix} \Sigma_1 & \Sigma_3^T \\ \star & -\Sigma_2 \end{bmatrix} < 0 \quad \text{or} \quad \begin{bmatrix} -\Sigma_2 & \Sigma_3 \\ \star & \Sigma_1 \end{bmatrix} < 0.$$

Lemma 2.1.1 (Jensen's inequality) [37] *For a continuous function $\{x(s)|s \in [a, b]\} \rightarrow \mathbb{R}^n$, if a positive definite matrix N satisfies, the following integral inequality holds:*

$$\frac{1}{b-a} \int_a^b x^T(s) N x(s) ds \geq \left(\int_a^b x(s) ds \right)^T N \left(\int_a^b x(s) ds \right). \quad (2.1.1)$$

Lemma 2.1.2 (A generalized free-weighting-matrix inequality) [38] *For a continuous function $\{x(s)|s \in [a, b]\} \rightarrow \mathbb{R}^n$, a given positive definite matrix Q , any matrices S and U with appropriate dimensions and the following integral inequality holds:*

$$\begin{aligned} - \int_a^b x^T(s) Q x(s) ds &\leq \text{Sym} \{ \omega_0^T S \omega_1 + \omega_0^T U \omega_2 \} \\ &\quad + (b-a) \omega_0^T \left(S Q^{-1} S^T + \frac{1}{3} U Q^{-1} U^T \right) \omega_0, \end{aligned} \quad (2.1.2)$$

where ω_0 is any vector, $\omega_1 = \int_a^b x(s) ds$, and $\omega_2 = - \int_a^b x(s) ds + \frac{2}{b-a} \int_a^b \int_s^b x(u) du ds$.

Lemma 2.1.3 (Finsler's lemma) [40] *Let $\zeta \in \mathbb{R}^n$, $\Pi \in \mathbb{S}^n$, and $M \in \mathbb{R}^{n \times m}$. The statements of equivalent follow:*

- (i) $\zeta^T \Pi \zeta < 0$, $\forall M \zeta = 0$, $\zeta \neq 0$,
- (ii) $(M^\perp)^T \Pi M^\perp < 0$,
- (iii) $\exists X \in \mathbb{R}^{m \times n} : \Pi + \text{Sym}\{XM\} < 0$.

Lemma 2.1.4 [39] *For each $s \in S$, let $S \subseteq \mathbb{R}^d$, $\Pi : S \rightarrow \mathbb{S}^n$, and $M : S \rightarrow \mathbb{R}^{m \times n}$. The statements of equivalent follow:*

- (i) $x^T \Pi(s)x < 0$, $\forall Mx = 0$, $x \neq 0$,
- (ii) $(\exists \mu(s) \in \mathbb{R}) : \Pi(s) - \mu(s)M^T(s)M(s) < 0$,
- (iii) *If there exists $X(s) \in \mathbb{R}^{n \times m}$, $\Pi(s) + X(s)M(s) + M^T(s)X^T(s) < 0$,*
- (iv) $(M^\perp(s))^T \Pi(s)M^\perp(s) < 0$.

Remark 1 Finsler's lemma [40] has been frequently utilized in previous researches. In [39], it is extended to the parameter-dependent systems for dealing with uncertain systems and contains proof of the extended Finsler's lemma 2.1.3. Lemma 2.1.4 is used to derive the inequality and understand the AZEA in the Theorems of this paper.

Lemma 2.1.5 [47] *Let $V(x(t))$ be a differentiable Lyapunov function for system with $V(x(0)) = 0$ and external inputs $\omega(t)$ satisfying $\omega^T(t)\omega(t) \leq \omega_M^2$. If there exists scalars $\alpha > 0$ and $\beta > 0$ such that $\dot{V}(x(t)) + \alpha V(x(t)) - \beta \omega^T(t)\omega(t) \leq 0$, then $V(x(t)) \leq \frac{\beta}{\alpha} \omega_M$.*

2.2 Stability criteria of a time-delay system

Let us consider a time-delay system as follows:

$$\dot{x}(t) = f(t, x_t), \quad t \geq t_0, \quad (2.2.1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector and the function $f : \mathbb{R}^+ \times \mathbb{C}_{n,h} \rightarrow \mathbb{R}^n$ is continuous and $f(t, 0) = 0$ for $\forall t \in \mathbb{R}$.

Let us define x_t as

$$x_t = x(t + \theta), \quad -h_M \leq \theta \leq 0. \quad (2.2.2)$$

The initial condition (2.2.1) is defined as

$$x_{t_0}(\theta) = \phi(\theta), \quad -h_M \leq \theta \leq 0. \quad (2.2.3)$$

It is assumed that the time-delayed system (2.2.1) with the initial condition (2.2.3) has a unique solution for $\phi \in \mathbb{C}_{n,h}$ and $x_{t_0} \in \mathbb{R}$.

Now, following is Lyapunov-Krasovskii's stability theorem.

Theorem 2.2.1 (Lyapunov-Krasovskii's Theorem) [2] *Suppose that the function f take bounded sets of $\mathbb{C}_{n,h}$ in bounded sets of $\mathbb{R}_n$, and $u, v, w : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ are continuous, non-decreasing functions satisfying $u(0) = v(0) = w(0) = 0$ and $u(s), v(s) > 0$ for $s > 0$. If there exists a continuous function $V : \mathbb{R} \times \mathbb{C}_{n,h} \rightarrow \mathbb{R}^+$ such that*

$$u(\|x\|) \leq V(t, x_t) \leq v(\|x_t\|_c). \quad (2.2.4)$$

The derivative of $V(t, x_t)$ along the solution of (2.2.1) and (2.2.3), defined as

$$\dot{V}(t, x_t) = \limsup_{s \rightarrow 0^+} \frac{1}{s} \left[V(t + s, x_{t+s}) - V(t, x_t) \right], \quad (2.2.5)$$

satisfies

$$\dot{V}(t, x_t) = -w(\|x(t)\|), \quad (2.2.6)$$

then the trivial solution $x = 0$ of the time-delay system in (2.2.1) and (2.2.3) is uniformly stable.

If $u(s) \rightarrow \infty$ as $s \rightarrow \infty$, the solutions of the time-delay system in (2.2.1) and (2.2.3) are uniformly bounded.

If $w(s) > 0$ for $s > 0$, the solutions $x = 0$ is uniformly asymptotically stable.

If, in addition,

$$\lim_{s \rightarrow 0} u(s) = \infty, \quad (2.2.7)$$

then it is globally asymptotically stable.

2.3 The model of neural networks

Neural networks mean the sets of neurons, basic units for a system. Each neuron generally is organized into layers, and each unit is interconnected between layers. The following Figure 2.3.1 helps the neural networks understand easily. The structure of neural networks consists of several layers, which are called input, hidden, and output layers. Among them, the hidden layer can be composed of several stages. The relation of layers makes the activation function in neural networks systems. The activation function is specifically explained by the following Figure 2.3.2. In Figure 2.3.2, inputs indicate the states of neurons. They have each weight, which includes the importance of synapse. Several inputs containing each weight are summed as a transfer function, and they are constructed the activation function by net input. The relationship between the activation function and neurons is simply expressed as $f = \sum_i w_i x_i$.

Science and engineering fields such as pattern recognition, secure communication, image processing, and associative memories have been applied to develop various industries in neural networks system. Therefore, the study of neural networks plays an important role and has a meaning in contributing to modern industries.

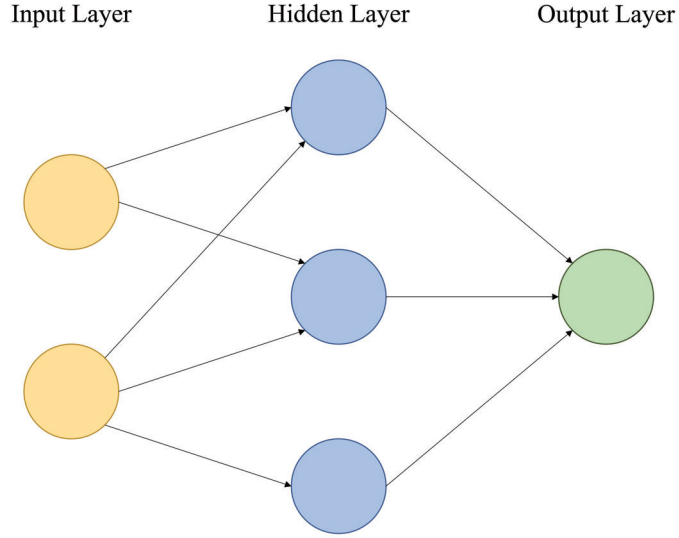


Figure 2.3.1: The structure of neural networks

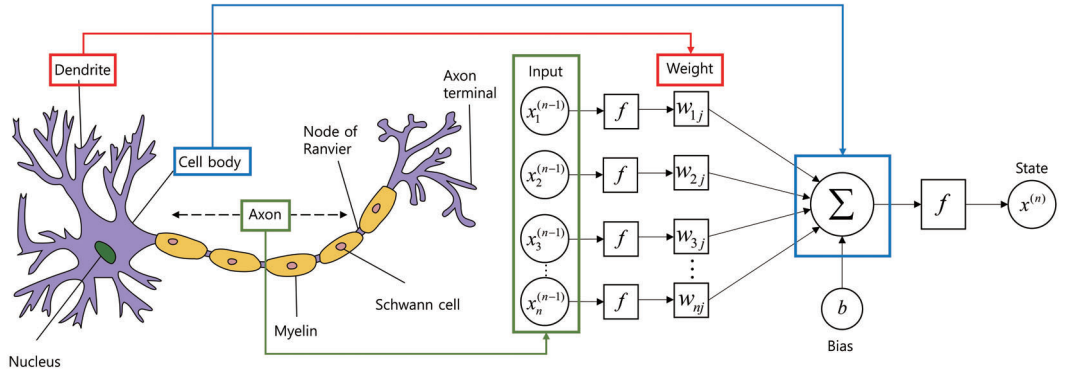


Figure 2.3.2: The structure of activation function

2.4 Markovian jump process

From the work [3], let $\mathbf{S} = 1, 2, \dots, N$ be an index set. The Markovian jump process $r(t), (t \geq 0)$ with values in the finite set $\mathbf{S}$, describes the switching between the

different modes and its evolution is governed by the following probability transitions:

$$P\{r_{t+\Delta t} = q | r_t = p\} = \begin{cases} \pi_{pq}\Delta t + o(\Delta t), & p \neq q \\ 1 + \pi_{pp}\Delta t + o(\Delta t), & p = q \end{cases} \quad (2.4.1)$$

where $\Delta t > 0$ and $\lim_{\Delta \rightarrow 0} \frac{o(\Delta t)}{\Delta t} = 0$. $\pi_{pq} \geq 0$ is the transition probability rate from mode p to mode q at time t and $\pi_{pp} = -\sum_{q=1, q \neq p}^N \pi_{pq}$.

To understand the Markovian jump process, the example of the status diagram is given as follows. In Figure 2.4.1, the transition matrix can be expressed $P = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix}$ with an index set $S = 1, 2$. The principles of the transition matrix mean the probability changed self or other system.

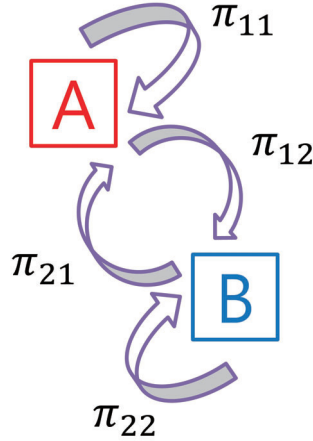


Figure 2.4.1: The state diagram of Markovian jump process

III. Neural networks

In this chapter, the problem of stability analysis is dealt with in a generalized neural networks system, and the estimation of the reachable set is applied in Markovian jump neural networks systems with time-varying delays. In Theorem 3.1.1-3.1.3, a stability analysis method with better results is suggested based on previous studies in NNTVDs. Comparison with the results of Corollary 3.1.3 also demonstrates the superiority and novelty of the LKFs proposed in section 3.1. Based on the principles of section 3.1, the estimation method of the reachable set is proposed in section 3.2. It aims to minimize the area of the reachable set while optimizing specific parameter values in Markovian jump NNVTs. Via various numerical examples, the superiority, effectiveness and necessity of the proposed methods are explained.

3.1 The stability analysis of neural networks

In this section, the stability analysis of a generalized NNTVDs is considered. The augmented vectors including the activation function-based integral terms, $\int_{t-h_M}^t f(W_2x(s))ds$ and $\int_{t-h_M}^t \int_s^t f(W_2x(u))duds$, are proposed. By constructing the augmented LKFs, a wider feasible region can be obtained. Especially, these terms are used for the double integral LKFs, which helps to get improved results than previous ones. Some mathematical techniques, such as zero equalities and AZEA, induce a great effect. These methods are critical techniques not only to eliminate the free variables with zero equalities and but also less conservative results. With the new LKFs considering the activation function-based terms and the generalized free-weighting-matrix inequality, stability conditions of a delay-dependent are derived for a generalized NNTVDs in Theorem 3.1.1. By including the positiveness of

the proposed LKFs, it can be deduced the improved results of feasible region in Theorem 3.1.2. Furthermore, by utilizing some mathematical techniques such as some zero equalities and AZEA, the following notions are proposed in Theorem 3.1.3. By comparing the results of Theorem 3.1.3 and Corollary 3.1.3, the proposed LKFs are proved their superiority. Finally, three numerical examples verify the validity and distinction of the proposed results in Theorems 3.1.1-3.1.3.

3.1.1 Problem Statements

Let us consider the following system for neural networks with time-varying delays

$$\dot{x}(t) = -Ax(t) + W_0f(W_2x(t)) + W_1f(W_2x(t - h(t))), \quad \forall t > 0, \quad (3.1.1)$$

where $x(t) \in \mathbb{R}^n$ is the neuron state vector, $f(\cdot) \in \mathbb{R}^n = [f_1(\cdot), f_2(\cdot), \dots, f_n(\cdot)]^T$ is the activation function of the neurons, $A \in \mathbb{R}^{n \times n}$ is a positive diagonal matrix, W_0, W_1 , and $W_2 \in \mathbb{R}^{n \times n}$ are the interconnection matrices between neurons. $h(t)$ is a continuous function which means the time-varying delays satisfying

$$0 \leq h(t) \leq h_M, \quad \dot{h}(t) \leq h_D, \quad (3.1.2)$$

where h_M is a positive constant and h_D is any scalar.

The activation function $f(\cdot)$ is satisfied with the following assumption.

Assumption 1 The activation functions $f_i(\cdot)$ ($i = 1, \dots, n$) with $f_i(0) = 0$ are considered continuous and assumed to be bounded. They are satisfied with the following condition:

$$k_i^- \leq \frac{f_i(b) - f_i(a)}{b - a} \leq k_i^+, \quad a \neq b \in \mathbb{R}, \quad (3.1.3)$$

where k_i^- and k_i^+ are real constants.

Remark 2 The concept of assumption 1 was explained from [36]. Activation function $f(\cdot)$ contains many activation conditions.

This paper aims to derive the delay-dependent stability analysis of the system (3.1.1) with time-varying delays. By constructing LKFs and utilizing some mathematical techniques, enhanced stability conditions will be derived in the forms of LMIs.

3.1.2 Main Results

This paper considers the main results using Lemmas 2.1.1-2.1.3 in neural networks with time-varying delays. To represent the main results, block entry matrices, $e_i (i = 1, 2, \dots, 18) \in \mathbb{R}^{18n \times n}$, are usually used to simply express matrices and vector states. For examples, $e_5 = [0_{n \times 4n}, I_n, 0_{n \times 13n}]^T$ and $e_{11} = [0_{n \times 10n}, I_n, 0_{n \times 7n}]^T$.

Some notations are explained below as

$$\begin{aligned}
K_p &= \text{diag}\{k_1^+, \dots, k_n^+\}, \quad K_m = \text{diag}\{k_1^-, \dots, k_n^-\}, \\
\zeta(t) &= \left[x^T(t), x^T(t-h(t)), x^T(t-h_M), \dot{x}^T(t), \dot{x}^T(t-h_M), \int_{t-h(t)}^t x^T(s)ds, \right. \\
&\quad \int_{t-h_M}^{t-h(t)} x^T(s)ds, \frac{1}{h(t)} \int_{t-h(t)}^t \int_s^t x^T(u)duds, \\
&\quad \frac{1}{h_M-h(t)} \int_{t-h_M}^{t-h(t)} \int_s^{t-h(t)} x^T(u)duds, \frac{1}{h(t)} \int_{t-h(t)}^t x^T(s)ds, \\
&\quad \frac{1}{h_M-h(t)} \int_{t-h_M}^{t-h(t)} x^T(s)ds, f^T(x(t)), f^T(x(t-h(t))), f^T(x(t-h_M)), \\
&\quad \int_{t-h(t)}^t f^T(W_2x(s))ds, \int_{t-h_M}^{t-h(t)} f^T(W_2x(s))ds, \\
&\quad \frac{1}{h(t)} \int_{t-h(t)}^t \int_s^t f^T(W_2x(u))duds, \\
&\quad \left. \frac{1}{h_M-h(t)} \int_{t-h_M}^{t-h(t)} \int_s^{t-h(t)} f^T(W_2x(u))duds \right]^T, \\
\beta(t) &= [x^T(t), \dot{x}^T(t), f^T(W_2x(t))]^T,
\end{aligned}$$

$$\zeta_1(t, s) = \left[x^T(t), x^T(t - h_M), \int_{t-h_M}^t x^T(s) ds, \int_{t-h_M}^t \int_s^t x^T(u) du ds, \right. \\ \left. \int_{t-h_M}^t f^T(W_2 x(s)) ds, \int_{t-h_M}^t \int_s^t f^T(W_2 x(u)) du ds \right]^T,$$

$$\zeta_2(t, s) = \left[\alpha^T(t), \int_s^t \alpha^T(u) du \right]^T,$$

$$\zeta_3(t, s) = \left[x^T(s), f^T(W_2 x(s)), \int_s^t \alpha^T(u) du, x^T(t - h_M) \right]^T,$$

$$Q_1^{aug} = Q + \begin{bmatrix} 0_n & P_1 & 0_n \\ \star & 0_n & 0_n \\ \star & \star & 0_n \end{bmatrix}, \quad Q_2^{aug} = Q + \begin{bmatrix} 0_n & P_2 & 0_n \\ \star & 0_n & 0_n \\ \star & \star & 0_n \end{bmatrix},$$

$$\Upsilon = [e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{12}, e_{13}, e_{14}, e_{15}, e_{16}, e_{17}, e_{18}],$$

$$\Xi_1[h(t)] = Sym \{ [e_1, e_3, e_6 + e_7, h(t)e_8 + (h_M - h(t))(e_9 + e_6), e_{15} + e_{16}, \\ h(t)e_{17} + (h_M - h(t))(e_{18} + e_{15})] \times R[e_4, e_5, e_1 - e_3, h_M e_1 - (e_6 + e_7), \\ e_{12} - e_{14}, h_M e_{12} - (e_{15} + e_{16})]^T \},$$

$$\Xi_2[h(t)] = [e_1, e_4, e_{12}, e_0, e_0, e_0] N[e_1, e_4, e_{12}, e_0, e_0, e_0]^T \\ - [e_3, e_5, e_{14}, e_6 + e_7, e_1 - e_3, e_{15} + e_{16}] N \\ \times [e_3, e_5, e_{14}, e_6 + e_7, e_1 - e_3, e_{15} + e_{16}]^T \\ + Sym \{ [e_6 + e_7, e_1 - e_3, e_{15} + e_{16}, h(t)e_8 + (h_M - h(t))(e_6 + e_9), \\ h_M e_1 - (e_6 + e_7), h(t)e_{17} + (h_M - h(t))(e_{15} + e_{18})] N \\ \times [e_0, e_0, e_0, e_1, e_4, e_{12}]^T \} \\ + Sym \{ [e_{12} - e_1 W_2^T K_m] \Lambda_1 W_2 e_4^T + [e_1 W_2^T K_p - e_{12}] \Delta_1 W_2 e_4^T \} \\ + Sym \{ [e_{14} - e_3 W_2^T K_m] \Lambda_2 W_2 e_5^T + [e_3 W_2^T K_p - e_{14}] \Delta_2 W_2 e_5^T \},$$

$$\Xi_{31} = [e_1, e_{12}, e_0, e_0, e_0, e_3] G[e_1, e_{12}, e_0, e_0, e_0, e_3]^T \\ - (1 - h_D) [e_2, e_{13}, e_6, e_1 - e_2, e_{15}, e_3] G[e_2, e_{13}, e_6, e_1 - e_2, e_{15}, e_3]^T,$$

$$\Xi_{32}[h(t)] = Sym \{ [e_6, e_{15}, h(t)e_8, h(t)e_1 - e_6, h(t)e_{17}, h(t)e_3] G \\ \times [e_0, e_0, e_1, e_4, e_{12}, e_5]^T \},$$

$$\begin{aligned}
\Xi_4 &= h_M [e_1, e_4, e_{12}] Q [e_1, e_4, e_{12}]^T, \\
\Xi_{41} &= Sym \left\{ S_1 [e_6, e_1 - e_2, e_{15}]^T \right. \\
&\quad \left. + U_1 [-e_6 + 2e_8, e_1 + e_2 - 2e_{10}, -e_{15} + 2e_{17}]^T \right\}, \\
\Xi_{42} &= Sym \left\{ S_2 [e_7, e_2 - e_3, e_{16}]^T \right. \\
&\quad \left. + U_2 [-e_7 + 2e_9, e_2 + e_3 - 2e_{11}, -e_{16} + 2e_{18}]^T \right\}, \\
\Xi_5 &= e_1 P_1 e_1^T - e_2 (P_1 - P_2) e_2^T - e_3 P_2 e_3^T, \\
\Xi_{6[h(t)]} &= Sym \left\{ (h(t)e_{10} - e_6) \Psi_1 \Upsilon^T \right\} + Sym \left\{ ((h_M - h(t)) e_{11} - e_7) \Psi_2 \Upsilon^T \right\}, \\
\Theta_{1[h(t)]} &= - \sum_{i=1}^3 Sym \left\{ (e_{11+i} - e_i W_2^T K_m) \left(\frac{h(t)}{h_M} J_i + \frac{h_M - h(t)}{h_M} L_i \right) \right. \\
&\quad \left. \times (e_{11+i} - e_i W_2^T K_p)^T \right\}, \\
\Theta_{2[h(t)]} &= - \sum_{j=1}^2 Sym \left\{ (e_{11+j} - e_{12+j} - (e_j - e_{1+j}) W_2^T K_m) \right. \\
&\quad \left. \left(\frac{h(t)}{h_M} J_{3+j} + \frac{h_M - h(t)}{h_M} L_{3+j} \right) (e_{11+j} - e_{12+j} - (e_j - e_{1+j}) W_2^T K_p)^T \right\}, \\
\Xi[h(t)] &= \Xi_{1[h(t)]} + \Xi_{2[h(t)]} + \Xi_{31} + \Xi_{32[h(t)]} + \Xi_4 + \Xi_{41} + \Xi_{42} + \Xi_5 + \Theta_{1[h(t)]} \\
&\quad + \Theta_{2[h(t)]}, \\
\Gamma &= -Ae_1^T - e_4^T + W_0 e_{12}^T + W_1 e_{13}^T.
\end{aligned} \tag{3.1.4}$$

Now, the following theorem is given by the first main result.

Theorem 3.1.1 *For given scalars h_M and h_D , system (3.1.1) is asymptotically stable for $0 \leq h(t) \leq h_M$ and $\dot{h}(t) \leq h_D$, if there exist matrices $R, G, N \in \mathbb{S}_+^{6n}$, $Q \in \mathbb{S}_+^{3n}$, $\Lambda_i, \Delta_i \in \mathbb{D}_+^n$, $P_i \in \mathbb{S}^n$, $S_i, U_i \in \mathbb{R}^{18n \times 3n}$, $\Psi_i \in \mathbb{R}^{n \times 18n}$ ($i = 1, 2$), $J_j \in \mathbb{D}_+^n$, $L_j \in \mathbb{D}_+^n$ ($j = 1, 2, \dots, 5$) satisfying the following LMIs:*

$$\begin{bmatrix}
(\Gamma^\perp)^T (\Xi_{[h_M]} + \Xi_{6[h_M]}) (\Gamma^\perp) (\Gamma^\perp)^T (h_M S_1) (\Gamma^\perp)^T (h_M U_1) \\
\star & -h_M Q_1^{aug} & 0_{3n} \\
\star & \star & -3h_M Q_1^{aug}
\end{bmatrix} < 0, \tag{3.1.5}$$

$$\begin{bmatrix} (\Gamma^\perp)^T (\Xi_{[0]} + \Xi_{6[0]}) (\Gamma^\perp) (\Gamma^\perp)^T (h_M S_2) (\Gamma^\perp)^T (h_M U_2) \\ \star & -h_M Q_2^{aug} & 0_{3n} \\ \star & \star & -3h_M Q_2^{aug} \end{bmatrix} < 0, \quad (3.1.6)$$

$$Q_1^{aug} \geq 0, \quad (3.1.7)$$

$$Q_2^{aug} \geq 0, \quad (3.1.8)$$

where the notations are defined as (3.1.4).

Proof Let us see the following LKFs:

$$V(t) = V_1 + V_2 + V_3 + V_4, \quad (3.1.9)$$

where

$$\begin{aligned} V_1 &= \zeta_1^T(t, s) R \zeta_1(t, s), \\ V_2 &= \int_{t-h_M}^t \zeta_2^T(t, s) N \zeta_2(t, s) ds \\ &\quad + 2 \sum_{i=1}^n \left(\lambda_{1,i} \int_0^{W_{2,i}x(t)} (f_i(s) - k_i^- s) ds + \delta_{1,i} \int_0^{W_{2,i}x(t)} (k_i^+ s - f_i(s)) ds \right) \\ &\quad + 2 \sum_{i=1}^n \left(\lambda_{2,i} \int_0^{W_{2,i}x(t-h_M)} (f_i(s) - k_i^- s) ds \right. \\ &\quad \left. + \delta_{2,i} \int_0^{W_{2,i}x(t-h_M)} (k_i^+ s - f_i(s)) ds \right), \\ V_3 &= \int_{t-h(t)}^t \zeta_3^T(t, s) G \zeta_3(t, s) ds, \\ V_4 &= \int_{t-h_M}^t \int_s^t \beta^T(u) Q \beta(u) du ds. \end{aligned}$$

The time derivatives of V_1 and V_2 are written as

$$\dot{V}_1 = 2\zeta_1^T(t, s) R \dot{\zeta}_1(t, s)$$

$$= \zeta^T(t) \Xi_1[h(t)] \zeta(t), \quad (3.1.10)$$

$$\begin{aligned} \dot{V}_2 &= \zeta_2^T(t, t) N \zeta_2(t, t) - \zeta_2^T(t, t - h_M) N \zeta_2(t, t - h_M) \\ &\quad + 2 \left[\begin{array}{c} \int_{t-h_M}^t \beta(s) ds \\ \int_{t-\lambda_M}^t \int_s^t \beta(u) du ds \end{array} \right]^T N \left[\begin{array}{c} 0_{3n \times 1} \\ \beta(t) \end{array} \right] \\ &\quad + 2 [f(W_2 x(t)) - K_m W_2 x(t)]^T \Lambda_1 W_2 \dot{x}(t) \\ &\quad + 2 [K_p W_2 x(t) - f(W_2 x(t))]^T \Delta_1 W_2 \dot{x}(t) \\ &\quad + 2 [f(W_2 x(t - h_M)) - K_m W_2 x(t - h_M)]^T \Lambda_2 W_2 \dot{x}(t - h_M) \\ &\quad + 2 [K_p W_2 x(t - h_M) - f(W_2 x(t - h_M))]^T \Delta_2 W_2 \dot{x}(t - h_M) \\ &= \zeta^T(t) \Xi_2[h(t)] \zeta(t). \end{aligned} \quad (3.1.11)$$

Calculating the upper bounds of the time derivatives of V_3 can lead to

$$\begin{aligned} \dot{V}_3 &= \zeta_3^T(t, t) G \zeta_3(t, t) - (1 - \dot{h}(t)) \zeta_3^T(t, t - h(t)) G \zeta_3(t, t - h(t)) \\ &\quad + \int_{t-h(t)}^t \frac{d}{dt} (\zeta_3^T(t, s) G \zeta_3(t, s)) ds \\ &\leq \zeta_3^T(t, t) G \zeta_3(t, t) - (1 - h_D) \zeta_3^T(t, t - h(t)) G \zeta_3(t, t - h(t)) \\ &\quad + \int_{t-h(t)}^t \frac{d}{dt} (\zeta_3^T(t, s) G \zeta_3(t, s)) ds \\ &= \zeta^T(t) (\Xi_{31} + \Xi_{32}[h(t)]) \zeta(t), \end{aligned} \quad (3.1.12)$$

since the condition satisfies $\dot{h}(t) \leq h_D$ in (3.1.2).

Calculating the time derivatives of V_4 is

$$\begin{aligned} \dot{V}_4 &= h_M \beta^T(t) Q \beta(t) - \int_{t-h_M}^t \beta^T(s) Q \beta(s) ds \\ &= \zeta^T(t) \Xi_4 \zeta(t) - \int_{t-h_M}^t \beta^T(s) Q \beta(s) ds. \end{aligned} \quad (3.1.13)$$

Two zero equalities are considered by the Newton-Leibniz formula as follows:

$$0 = x^T(t) P_1 x(t) - x^T(t - h(t)) P_1 x(t - h(t))$$

$$-2 \int_{t-h(t)}^t x^T(s) P_1 \dot{x}(s) ds, \quad (3.1.14)$$

$$0 = x^T(t-h(t)) P_2 x(t-h(t)) - x^T(t-h_M) P_2 x(t-h_M) \\ -2 \int_{t-h_M}^{t-h(t)} x^T(s) P_2 \dot{x}(s) ds. \quad (3.1.15)$$

The sum of (3.1.14) and (3.1.15) can lead to

$$0 = \zeta^T(t) \Xi_5 \zeta(t) - 2 \int_{t-h(t)}^t x^T(s) P_1 \dot{x}(s) ds \\ -2 \int_{t-h_M}^{t-h(t)} x^T(s) P_2 \dot{x}(s) ds. \quad (3.1.16)$$

By adding (3.1.16) into (3.1.13), it can be

$$\dot{V}_4 = \zeta^T(t) (\Xi_4 + \Xi_5) \zeta(t) - \int_{t-h(t)}^t \beta^T(s) \left(Q + \begin{bmatrix} 0_n & P_1 & 0_n \\ \star & 0_n & 0_n \\ \star & \star & 0_n \end{bmatrix} \right) \beta(s) ds \\ - \int_{t-h_M}^{t-h(t)} \beta^T(s) \left(Q + \begin{bmatrix} 0_n & P_2 & 0_n \\ \star & 0_n & 0_n \\ \star & \star & 0_n \end{bmatrix} \right) \beta(s) ds \\ = \zeta^T(t) (\Xi_4 + \Xi_5) \zeta(t) - \int_{t-h(t)}^t \beta^T(s) Q_1^{aug} \beta(s) ds \\ - \int_{t-h_M}^{t-h(t)} \beta^T(s) Q_2^{aug} \beta(s) ds, \quad (3.1.17)$$

where Q_1^{aug} and Q_2^{aug} are defined as (3.1.4).

From Lemma 2.1.2, the integral terms in (3.1.17) get the upper bounds as

$$- \int_{t-h(t)}^t \beta^T(s) Q_1^{aug} \beta(s) ds \\ \leq \text{Sym} \{ \zeta^T(t) S_1 \omega_1(t) + \zeta^T(t) U_1 \omega_2(t) \} \\ + h(t) \zeta^T(t) \left(S_1 (Q_1^{aug})^{-1} S_1^T + \frac{1}{3} U_1 (Q_1^{aug})^{-1} U_1^T \right) \zeta(t), \quad (3.1.18) \\ - \int_{t-h_M}^{t-h(t)} \beta^T(s) Q_2^{aug} \beta(s) ds$$

$$\begin{aligned} &\leq \text{Sym} \{ \zeta^T(t) S_2 \omega_3(t) + \zeta^T(t) U_2 \omega_4(t) \} \\ &\quad + (h_M - h(t)) \zeta^T(t) \left(S_2 (Q_2^{aug})^{-1} S_2^T + \frac{1}{3} U_2 (Q_2^{aug})^{-1} U_2^T \right) \zeta(t), \quad (3.1.19) \end{aligned}$$

where

$$\begin{aligned} \omega_1(t) &= \int_{t-h(t)}^t \beta(s) ds = [e_6, e_1 - e_2, e_{15}]^T \zeta(t), \\ \omega_2(t) &= - \int_{t-h(t)}^t \beta(s) ds + \frac{2}{h(t)} \int_{t-h(t)}^t \int_s^t \beta(u) du ds \\ &= [-e_6 + 2e_8, e_1 + e_2 - 2e_{10}, -e_{15} + 2e_{17}]^T \zeta(t), \\ \omega_3(t) &= \int_{t-h_M}^{t-h(t)} \beta(s) ds = [e_7, e_2 - e_3, e_{16}]^T \zeta(t), \\ \omega_4(t) &= - \int_{t-h_M}^{t-h(t)} \beta(s) ds + \frac{2}{h_M - h(t)} \int_{t-h_M}^{t-h(t)} \int_s^{t-h(t)} \beta(u) du ds \\ &= [-e_7 + 2e_9, e_2 + e_3 - 2e_{11}, -e_{16} + 2e_{18}]^T \zeta(t). \end{aligned}$$

The upper bound of $\dot{V}_4$ derived using (3.1.13)-(3.1.19) is as follows:

$$\begin{aligned} \dot{V}_4 &\leq \zeta^T(t) (\Xi_4 + \Xi_{41} + \Xi_{42} + \Xi_5) \zeta(t) \\ &\quad + h(t) \zeta^T(t) \left\{ S_1 (Q_1^{aug})^{-1} S_1^T + \frac{1}{3} U_1 (Q_1^{aug})^{-1} U_1^T \right\} \zeta(t) \\ &\quad + (h_M - h(t)) \zeta^T(t) \left\{ S_2 (Q_2^{aug})^{-1} S_2^T + \frac{1}{3} U_2 (Q_2^{aug})^{-1} U_2^T \right\} \zeta(t). \quad (3.1.20) \end{aligned}$$

In the component association of $\zeta(t)$, two zero equalities are derived:

$$0 = \zeta^T(t) \text{Sym} \{ (h(t) e_{10} - e_6) \Psi_1 \Upsilon^T \} \zeta(t), \quad (3.1.21)$$

$$0 = \zeta^T(t) \text{Sym} \{ ((h_M - h(t)) e_{11} - e_7) \Psi_2 \Upsilon^T \} \zeta(t), \quad (3.1.22)$$

where $\Psi_i \in \mathbb{R}^{n \times 18n}$ ($i = 1, 2$) are any matrices.

The sum of (3.1.21) and (3.1.22) can lead to

$$0 = \zeta^T(t) \Xi_{6[h(t)]} \zeta(t). \quad (3.1.23)$$

From Assumption 1 with $a = 0$, for $J_j, L_j \in \mathbf{D}_+^n$ ($j = 1, 2, 3$), the derived inequality is satisfied:

$$\zeta^T(t) \Theta_{1[h(t)]} \zeta(t) \geq 0, \quad (3.1.24)$$

and with $a \neq 0$, for $J_j, L_j \in \mathbf{D}_+^n$ ($j = 4, 5$), the derived inequality is also satisfied:

$$\zeta^T(t) \Theta_{2[h(t)]} \zeta(t) \geq 0, \quad (3.1.25)$$

The proof of the inequalities (3.1.24) and (3.1.25) was specifically explained in [16].

From (3.1.11)-(3.1.25), an upper bound of $\dot{V}(t)$ can be written as

$$\begin{aligned} \dot{V}(t) \leq & \zeta^T(t) \left[\Xi_{[h(t)]} + \Xi_{6[h(t)]} + h(t) \left\{ S_1(Q_1^{aug})^{-1} S_1^T + \frac{1}{3} U_1(Q_1^{aug})^{-1} U_1^T \right\} \right. \\ & \left. + (h_M - h(t)) \left\{ S_2(Q_2^{aug})^{-1} S_2^T + \frac{1}{3} U_2(Q_2^{aug})^{-1} U_2^T \right\} \right] \zeta(t). \end{aligned} \quad (3.1.26)$$

A stability condition in the system (3.1.1) can be

$$\begin{aligned} & \zeta^T(t) \left[\Xi_{[h(t)]} + \Xi_{6[h(t)]} + h(t) \left\{ S_1(Q_1^{aug})^{-1} S_1^T + \frac{1}{3} U_1(Q_1^{aug})^{-1} U_1^T \right\} \right. \\ & \left. + (h_M - h(t)) \left\{ S_2(Q_2^{aug})^{-1} S_2^T + \frac{1}{3} U_2(Q_2^{aug})^{-1} U_2^T \right\} \right] \zeta(t) < 0, \end{aligned} \quad (3.1.27)$$

subject to $0 = \Gamma \zeta(t)$.

From (i) and (ii) of Lemma 2.1.3, the following inequality satisfies:

$$\begin{aligned} & \left(\Gamma^\perp \right)^T \left[\Xi_{[h(t)]} + \Xi_{6[h(t)]} + h(t) \left\{ S_1(Q_1^{aug})^{-1} S_1^T + \frac{1}{3} U_1(Q_1^{aug})^{-1} U_1^T \right\} \right. \\ & \left. + (h_M - h(t)) \left\{ S_2(Q_2^{aug})^{-1} S_2^T + \frac{1}{3} U_2(Q_2^{aug})^{-1} U_2^T \right\} \right] \left(\Gamma^\perp \right) < 0. \end{aligned} \quad (3.1.28)$$

The inequality (3.1.28) is satisfied, if the following two inequalities hold

$$\begin{aligned} & \left(\Gamma^\perp \right)^T \left[\Xi_{[h_M]} + \Xi_{6[h_M]} \right. \\ & \left. + h_M \left\{ S_1(Q_1^{aug})^{-1} S_1^T + \frac{1}{3} U_1(Q_1^{aug})^{-1} U_1^T \right\} \right] \left(\Gamma^\perp \right) < 0 \end{aligned} \quad (3.1.29)$$

and

$$\begin{aligned} & \left(\Gamma^\perp \right)^T \left[\Xi_{[0]} + \Xi_{6[0]} \right. \\ & \left. + h_M \left\{ S_2(Q_2^{aug})^{-1} S_2^T + \frac{1}{3} U_2(Q_2^{aug})^{-1} U_2^T \right\} \right] \left(\Gamma^\perp \right) < 0. \end{aligned} \quad (3.1.30)$$

since it relies on $h(t)$ with the condition of $0 \leq h(t) \leq h_M$. Therefore, the stability conditions for Generalized NNTVDs (3.1.1) satisfy with (3.1.5)-(3.1.8), since the inequalities (3.1.29) and (3.1.30) can be developed by Schur's complements. Thus, it is asymptotically stable for $0 \leq h(t) \leq h_M$ and $\dot{h}(t) \leq h_D$. This completes our proof. $\blacksquare$

Remark 3 It was proved that inserting inequalities containing the activation function condition (3.1.3) provides enhanced stability criteria for Generalized NNTVDs in [10]. Like [28], $\frac{h(t)}{h_M} J_i + \frac{h_M - h(t)}{h_M} L_i$ ($i = 1, \dots, 5$) are always positive definite since the time-varying delay $0 \leq h(t) \leq h_M$ relies on $h(t)$.

To represent the following Theorem 3.1.2, $\tilde{e}_i$ ($i = 1, \dots, 6$) $\in \mathbb{R}^{6n \times n}$, are used to simply express matrices and vector states. If $N > 0$ satisfies, the lower bound of $V(t)$ follows as:

$$V(t) > \zeta_1^T(t, s) R \zeta_1(t, s) + \int_{t-h_M}^t \zeta_2^T(t, s) N \zeta_2(t, s) ds \quad (3.1.31)$$

By Lemma 2.1.1, the lower bound of $\int_{t-h_M}^t \zeta_2^T(t, s) N \zeta_2(t, s) ds$ can be easily obtained as

$$\begin{aligned} & \int_{t-h_M}^t \zeta_2^T(t, s) N \zeta_2(t, s) ds \\ & \geq \frac{1}{h_M} \left(\int_{t-h_M}^t \zeta_2(t, s) ds \right)^T N \left(\int_{t-h_M}^t \zeta_2(t, s) ds \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{h_M} \begin{bmatrix} \int_{t-h_M}^t x(s) ds \\ x(t) - x(t-h_M) \\ \int_{t-h_M}^t f(W_2 x(s)) ds \\ \int_{t-h_M}^t \int_s^t x(u) du ds \\ h_M x(t) - \int_{t-h_M}^t x(s) ds \\ \int_{t-h_M}^t \int_s^t f(W_2 x(u)) du ds \end{bmatrix}^T N \begin{bmatrix} \int_{t-h_M}^t x(s) ds \\ x(t) - x(t-h_M) \\ \int_{t-h_M}^t f(W_2 x(s)) ds \\ \int_{t-h_M}^t \int_s^t x(u) du ds \\ h_M x(t) - \int_{t-h_M}^t x(s) ds \\ \int_{t-h_M}^t \int_s^t f(W_2 x(u)) du ds \end{bmatrix} \\
&= \frac{1}{h_M} \zeta_1^T(t, s) [\tilde{e}_3, \tilde{e}_1 - \tilde{e}_2, \tilde{e}_5, \tilde{e}_4, h_M \tilde{e}_1 - \tilde{e}_3, \tilde{e}_6] N \\
&\quad \times [\tilde{e}_3, \tilde{e}_1 - \tilde{e}_2, \tilde{e}_5, \tilde{e}_4, h_M \tilde{e}_1 - \tilde{e}_3, \tilde{e}_6]^T \zeta_1(t, s)
\end{aligned} \tag{3.1.32}$$

Therefore, if the following inequality holds,

$$\begin{aligned}
&R + \frac{1}{h_M} [\tilde{e}_3, \tilde{e}_1 - \tilde{e}_2, \tilde{e}_5, \tilde{e}_4, h_M \tilde{e}_1 - \tilde{e}_3, \tilde{e}_6] N \\
&\quad \times [\tilde{e}_3, \tilde{e}_1 - \tilde{e}_2, \tilde{e}_5, \tilde{e}_4, h_M \tilde{e}_1 - \tilde{e}_3, \tilde{e}_6]^T > 0
\end{aligned} \tag{3.1.33}$$

then the lower bound of $V(t)$ can be guaranteed to be positive.

The following Theorem 3.1.2 shows the effectiveness of positiveness based on Theorem 3.1.1.

Theorem 3.1.2 *For given scalars h_M and h_D , system (3.1.1) is asymptotically stable for $0 \leq h(t) \leq h_M$ and $\dot{h}(t) \leq h_D$, if there exist matrices $R \in \mathbb{S}^{6n}$, $G, N \in \mathbb{S}_+^{6n}$, $Q \in \mathbb{S}_+^{3n}$, $\Lambda_i, \Delta_i \in \mathbb{D}_+^n$, $P_i \in \mathbb{S}^n$, $S_i, U_i \in \mathbb{R}^{18n \times 3n}$, $\Psi_i \in \mathbb{R}^{n \times 18n}$ ($i = 1, 2$), $J_j \in \mathbb{D}_+^n$, $L_j \in \mathbb{D}_+^n$ ($j = 1, 2, \dots, 5$) satisfying the LMIs (3.1.5)-(3.1.8), and (3.1.33).*

Proof The augmented LKFs are the same in Theorem 3.1.1 (3.1.9) and has the lower bound $V(t)$ like (3.1.31). The other proof and notations are the same as the previous theorem, so it is omitted. ■

Remark 4 The lower bound of $V(t)$ can be held by integrating some functionals of $V(t)$ in Theorem 3.1.1. The positiveness of the matrix R replaces from $R > 0$ to (3.1.33). The proposed lower bound of $V(t)$ guarantees less conservatism in stability analysis.

The following Theorem 3.1.3 shows an improved result than the results of Theorem 3.1.2.

Theorem 3.1.3 For given scalars h_M and h_D , system (3.1.1) is asymptotically stable for $0 \leq h(t) \leq h_M$ and $\dot{h}(t) \leq h_D$, if there exist matrices $R \in \mathbb{S}^{6n}$, $G, N \in \mathbb{S}_+^{6n}$, $Q \in \mathbb{S}_+^{3n}$, $\Lambda_i, \Delta_i \in \mathbb{D}_+^n$, $P_i \in \mathbb{S}^n$, $S_i, U_i \in \mathbb{R}^{18n \times 3n}$ ($i = 1, 2$), $J_j \in \mathbb{D}_+^n$, $L_j \in \mathbb{D}_+^n$ ($j = 1, 2, \dots, 5$) satisfying (3.1.7), (3.1.8), (3.1.33), and the following LMIs :

$$\begin{bmatrix} \left(\Gamma_{[h_M]}^\perp\right)^T \Xi_{[h_M]} \left(\Gamma_{[h_M]}^\perp\right) \left(\Gamma_{[h_M]}^\perp\right)^T (h_M S_1) \left(\Gamma_{[h_M]}^\perp\right)^T (h_M U_1) \\ \star & -h_M Q_1^{aug} & 0_{3n} \\ \star & \star & -3h_M Q_1^{aug} \end{bmatrix} < 0, \quad (3.1.34)$$

$$\begin{bmatrix} \left(\Gamma_{[0]}^\perp\right)^T \Xi_{[0]} \left(\Gamma_{[0]}^\perp\right) \left(\Gamma_{[0]}^\perp\right)^T (h_M S_2) \left(\Gamma_{[0]}^\perp\right)^T (h_M U_2) \\ \star & -h_M Q_2^{aug} & 0_{3n} \\ \star & \star & -3h_M Q_2^{aug} \end{bmatrix} < 0, \quad (3.1.35)$$

where

$$\Gamma_{[h(t)]} = \begin{bmatrix} \Gamma \\ h(t)e_{10}^T - e_6^T \\ (h_M - h(t))e_{11}^T - e_7^T \end{bmatrix}.$$

The same notations are used, like Theorems 3.1.1 and 3.1.2.

Proof Let us see the stability condition of the LKF's for a system (3.1.1). It can be written by

$$\begin{aligned} \dot{V}(t) &\leq \zeta^T(t) \left[\Xi_{[h(t)]} + h(t) \left\{ S_1(Q_1^{aug})^{-1} S_1^T + \frac{1}{3} U_1(Q_1^{aug})^{-1} U_1^T \right\} \right. \\ &\quad \left. + (h_M - h(t)) \left\{ S_2(Q_2^{aug})^{-1} S_2^T + \frac{1}{3} U_2(Q_2^{aug})^{-1} U_2^T \right\} \right] \zeta(t) < 0, \\ &\text{subject to } 0 = \Gamma_{[h(t)]} \zeta(t). \end{aligned} \quad (3.1.36)$$

The inequality (3.1.36) can be developed by Lemma 2.1.3, and it is equivalent to

$$\begin{aligned} &\left(\Gamma_{[h(t)]}^\perp \right)^T \left[\Xi_{[h(t)]} + h(t) \left\{ S_1(Q_1^{aug})^{-1} S_1^T + \frac{1}{3} U_1(Q_1^{aug})^{-1} U_1^T \right\} \right. \\ &\quad \left. + (h_M - h(t)) \left\{ S_2(Q_2^{aug})^{-1} S_2^T + \frac{1}{3} U_2(Q_2^{aug})^{-1} U_2^T \right\} \right] \left(\Gamma_{[h(t)]}^\perp \right) < 0, \end{aligned} \quad (3.1.37)$$

where $\Xi_{[h(t)]}$ is the same notation in Theorem 3.1.1.

The inequality (3.1.37) is satisfied, if the following two inequalities hold

$$\begin{aligned} &\left(\Gamma_{[h_M]}^\perp \right)^T \left[\Xi_{[h_M]} + h_M \left\{ S_1(Q_1^{aug})^{-1} S_1^T \right. \right. \\ &\quad \left. \left. + \frac{1}{3} U_1(Q_1^{aug})^{-1} U_1^T \right\} \right] \left(\Gamma_{[h_M]}^\perp \right) < 0, \end{aligned} \quad (3.1.38)$$

$$\begin{aligned} &\left(\Gamma_{[0]}^\perp \right)^T \left[\Xi_{[0]} + h_M \left\{ S_2(Q_2^{aug})^{-1} S_2^T \right. \right. \\ &\quad \left. \left. + \frac{1}{3} U_2(Q_2^{aug})^{-1} U_2^T \right\} \right] \left(\Gamma_{[0]}^\perp \right) < 0, \end{aligned} \quad (3.1.39)$$

since the above inequality relies on $h(t)$ with the condition of $0 \leq h(t) \leq h_M$. Since the other proof is the same as Theorem 3.1.1, it can be omitted. $\blacksquare$

Remark 5 Zero equalities have some drawbacks, which increase the calculating costs due to the free variables $\Psi_i \in \mathbb{R}^{n \times 18n}$ ($i = 1, 2$). For solving these problems, the AZEA was proposed in [35]. All the zero equalities such that $0 = (-Ae_1^T - e_4^T + W_0 e_{12}^T$

$+W_1 e_{13}^T) \zeta(t)$, $0 = (h(t)e_{10} - e_6)\zeta(t)$, $0 = ((h_M - h(t))e_{11} - e_7)\zeta(t)$ in Theorem 3.1.1 are incorporated in $0 = \Gamma_{[h(t)]}\zeta(t)$. By utilizing the AZEA, Lemma 2.1.3 was applied to (3.1.36). Consequently, it is possible not only to eliminate the free variables $\Psi_i \in \mathbb{R}^{n \times 18n}$ ($i = 1, 2$), but also to guarantee less conservatism in stability criteria with improved maximum upper bounds of time-varying delays.

Remark 6 The proposed LKFs in previous researches such as [12] and [26] have not focused on and been not commonly used the activation function-based terms in NNTVDs. It can be expected the improved delay bounds than previous papers by including the activation function-based terms such as $\int_{t-h_M}^t f(W_2 x(s)) ds$ and $\int_{t-h_M}^t \int_s^t f(W_2 x(u)) duds$. To prove the effectiveness of the proposed LKFs above, it is suggested the other LKFs except for the vectors of the activation function-based terms in Corollary 3.1.3. By comparing the results in three examples, the effectiveness of the proposed LKFs will be verified.

The following Corollary 3.1.3 shows for comparison how much the activation function-based terms are improved results.

The notations for understanding the main results in Corollary 3.1.3 are defined as

$$\begin{aligned} \hat{\zeta}(t) = & \left[x^T(t), x^T(t-h(t)), x^T(t-h_M), \dot{x}^T(t), \dot{x}^T(t-h_M), \int_{t-h(t)}^t x^T(s) ds, \right. \\ & \int_{t-h_M}^{t-h(t)} x^T(s) ds, \frac{1}{h(t)} \int_{t-h(t)}^t \int_s^t x^T(u) duds, \\ & \frac{1}{h_M - h(t)} \int_{t-h_M}^{t-h(t)} \int_s^{t-h(t)} x^T(u) duds, \\ & \left. \frac{1}{h(t)} \int_{t-h(t)}^t x^T(s) ds, \frac{1}{h_M - h(t)} \int_{t-h_M}^{t-h(t)} x^T(s) ds, \right] \end{aligned}$$

$$\begin{aligned}
& \left. f^T(x(t)), f^T(x(t-h(t))), f^T(x(t-h_M)) \right]^T, \\
& \hat{e}_i = [0_{n \times (i-1)n}, I_n, 0_{n \times (14-i)n}]^T, \quad i = 1, 2, \dots, 14, \\
& \hat{\tilde{e}}_j = [0_{n \times (j-1)n}, I_n, 0_{n \times (4-j)n}]^T, \quad j = 1, 2, 3, 4, \\
& \hat{\beta}(t) = [x^T(t), \dot{x}^T(t)]^T, \\
& \hat{\zeta}_1(t, s) = \left[x^T(t), x^T(t-h_M), \int_{t-h_M}^t x^T(s) ds, \int_{t-h_M}^t \int_s^t x^T(u) du ds \right]^T, \\
& \hat{\zeta}_2(t, s) = \left[\hat{\beta}^T(t), \int_s^t \hat{\beta}^T(u) du \right]^T, \quad \hat{\zeta}_3(t, s) = \left[x^T(s), \int_s^t \hat{\beta}^T(u) du, x^T(t-h_M) \right]^T, \\
& \hat{Q}_1^{aug} = \hat{Q} + \begin{bmatrix} 0_n & P_1 \\ \star & 0_n \end{bmatrix}, \quad \hat{Q}_2^{aug} = \hat{Q} + \begin{bmatrix} 0_n & P_2 \\ \star & 0_n \end{bmatrix}, \\
& \hat{\Xi}_1[h(t)] = Sym \left\{ [\hat{e}_1, \hat{e}_3, \hat{e}_6 + \hat{e}_7, h(t)\hat{e}_8 + (h_M - h(t))(\hat{e}_9 + \hat{e}_6)] \hat{R} \right. \\
& \quad \left. \times [\hat{e}_4, \hat{e}_5, \hat{e}_1 - \hat{e}_3, h_M\hat{e}_1 - (\hat{e}_6 + \hat{e}_7)]^T \right\}, \\
& \hat{\Xi}_2[h(t)] = [\hat{e}_1, \hat{e}_4, \hat{e}_0, \hat{e}_0] \hat{N} [\hat{e}_1, \hat{e}_4, \hat{e}_0, \hat{e}_0]^T \\
& \quad - [\hat{e}_3, \hat{e}_5, \hat{e}_6 + \hat{e}_7, \hat{e}_1 - \hat{e}_3] \hat{N} [\hat{e}_3, \hat{e}_5, \hat{e}_6 + \hat{e}_7, \hat{e}_1 - \hat{e}_3]^T \\
& \quad + Sym \left\{ [\hat{e}_6 + \hat{e}_7, \hat{e}_1 - \hat{e}_3, h(t)\hat{e}_8 + (h_M - h(t))(\hat{e}_6 + \hat{e}_9), \right. \\
& \quad \left. h_M\hat{e}_1 - (\hat{e}_6 + \hat{e}_7)] \hat{N} \right. \\
& \quad \left. \times [\hat{e}_0, \hat{e}_0, \hat{e}_1, \hat{e}_4]^T \right\} \\
& \quad + Sym \left\{ [\hat{e}_{12} - \hat{e}_1 W_2^T K_m] \hat{\Lambda}_1 W_2 \hat{e}_4^T + [\hat{e}_1 W_2^T K_p - \hat{e}_{12}] \hat{\Delta}_1 W_2 \hat{e}_4^T \right\} \\
& \quad + Sym \left\{ [\hat{e}_{14} - \hat{e}_3 W_2^T K_m] \hat{\Lambda}_2 W_2 \hat{e}_5^T + [\hat{e}_3 W_2^T K_p - \hat{e}_{14}] \hat{\Delta}_2 W_2 \hat{e}_5^T \right\}, \\
& \hat{\Xi}_{31} = [\hat{e}_1, \hat{e}_0, \hat{e}_0, \hat{e}_3] \hat{G} [\hat{e}_1, \hat{e}_0, \hat{e}_0, \hat{e}_3]^T \\
& \quad - (1 - h_D) [\hat{e}_2, \hat{e}_6, \hat{e}_1 - \hat{e}_2, \hat{e}_3] \hat{G} [\hat{e}_2, \hat{e}_6, \hat{e}_1 - \hat{e}_2, \hat{e}_3]^T, \\
& \hat{\Xi}_{32}[h(t)] = Sym \left\{ [\hat{e}_6, h(t)\hat{e}_8, h(t)\hat{e}_1 - \hat{e}_6, h(t)\hat{e}_3] \hat{G} [\hat{e}_0, \hat{e}_1, \hat{e}_4, \hat{e}_5]^T \right\}, \\
& \hat{\Xi}_4 = h_M [\hat{e}_1, \hat{e}_4] \hat{Q} [\hat{e}_1, \hat{e}_4]^T, \\
& \hat{\Xi}_{41} = Sym \left\{ \hat{S}_1 [\hat{e}_6, \hat{e}_1 - \hat{e}_2]^T + \hat{U}_1 [-\hat{e}_6 + 2\hat{e}_8, \hat{e}_1 + \hat{e}_2 - 2\hat{e}_{10}]^T \right\}, \\
& \hat{\Xi}_{42} = Sym \left\{ \hat{S}_2 [\hat{e}_7, \hat{e}_2 - \hat{e}_3]^T + \hat{U}_2 [-\hat{e}_7 + 2\hat{e}_9, \hat{e}_2 + \hat{e}_3 - 2\hat{e}_{11}]^T \right\},
\end{aligned}$$

$$\begin{aligned}
\hat{\Xi}_5 &= \hat{e}_1 \hat{P}_1 \hat{e}_1^T - \hat{e}_2 \left(\hat{P}_1 - \hat{P}_2 \right) \hat{e}_2^T - \hat{e}_3 \hat{P}_2 \hat{e}_3^T, \\
\hat{\Theta}_{1[h(t)]} &= - \sum_{i=1}^3 \text{Sym} \left\{ \left(\hat{e}_{11+i} - \hat{e}_i W_2^T K_m \right) \left(\frac{h(t)}{h_M} \hat{H}_i + \frac{h_M - h(t)}{h_M} \hat{F}_i \right) \right. \\
&\quad \left. \times \left(\hat{e}_{11+i} - \hat{e}_i W_2^T K_p \right)^T \right\}, \\
\hat{\Theta}_{2[h(t)]} &= - \sum_{j=1}^2 \text{Sym} \left\{ \left(\hat{e}_{11+j} - \hat{e}_{12+j} - (\hat{e}_j - \hat{e}_{1+j}) W_2^T K_m \right) \right. \\
&\quad \times \left(\frac{h(t)}{h_M} \hat{H}_{3+j} + \frac{h_M - h(t)}{h_M} \hat{F}_{3+j} \right) \\
&\quad \left. \times \left(\hat{e}_{11+j} - \hat{e}_{12+j} - (\hat{e}_j - \hat{e}_{1+j}) W_2^T K_p \right)^T \right\}, \\
\hat{\Xi}_{[h(t)]} &= \hat{\Xi}_{1[h(t)]} + \hat{\Xi}_{2[h(t)]} + \hat{\Xi}_{31} + \hat{\Xi}_{32[h(t)]} + \hat{\Xi}_4 + \hat{\Xi}_{41} + \hat{\Xi}_{42} + \hat{\Xi}_5 \\
&\quad + \hat{\Theta}_{1[h(t)]} + \hat{\Theta}_{2[h(t)]}, \\
\hat{\Gamma} &= -A \hat{e}_1^T - \hat{e}_4^T + W_0 \hat{e}_{12}^T + W_1 \hat{e}_{13}^T.
\end{aligned} \tag{3.1.40}$$

Corollary 3.1.3 *For given scalars h_M and h_D , system (3.1.1) is asymptotically stable for $0 \leq h(t) \leq h_M$ and $\dot{h}(t) \leq h_D$, if there exist matrices $\hat{R} \in \mathbb{S}^{4n}$, $\hat{G}$, $\hat{N} \in \mathbb{S}_+^{4n}$, $\hat{Q} \in \mathbb{S}_+^{2n}$, $\hat{\Lambda}_i$, $\hat{\Delta}_i \in \mathbb{D}_+^n$, $\hat{P}_i \in \mathbb{S}^n$, $\hat{S}_i$, $\hat{U}_i \in \mathbb{R}^{14n \times 2n}$ ($i = 1, 2$), $\hat{J}_j \in \mathbb{D}_+^n$, $\hat{L}_j \in \mathbb{D}_+^n$ ($j = 1, 2, \dots, 5$) satisfying the following LMIs:*

$$\begin{bmatrix}
\left(\hat{\Gamma}_{[h_M]}^\perp \right)^T \hat{\Xi}_{[h_M]} \left(\hat{\Gamma}_{[h_M]}^\perp \right) \left(\hat{\Gamma}_{[h_M]}^\perp \right)^T \left(h_M \hat{S}_1 \right) \left(\hat{\Gamma}_{[h_M]}^\perp \right)^T \left(h_M \hat{U}_1 \right) \\
\star & -h_M \hat{Q}_1^{aug} & 0_{2n} \\
\star & \star & -3h_M \hat{Q}_1^{aug}
\end{bmatrix} < 0 \tag{3.1.41}$$

$$\begin{bmatrix}
\left(\hat{\Gamma}_{[0]}^\perp \right)^T \hat{\Xi}_{[0]} \left(\hat{\Gamma}_{[0]}^\perp \right) \left(\hat{\Gamma}_{[0]}^\perp \right)^T \left(h_M \hat{S}_2 \right) \left(\hat{\Gamma}_{[0]}^\perp \right)^T \left(h_M \hat{U}_2 \right) \\
\star & -h_M \hat{Q}_2^{aug} & 0_{2n} \\
\star & \star & -3h_M \hat{Q}_2^{aug}
\end{bmatrix} < 0, \tag{3.1.42}$$

$$\hat{Q}_1^{aug} \geq 0, \tag{3.1.43}$$

$$\hat{Q}_2^{aug} \geq 0, \quad (3.1.44)$$

$$\begin{aligned} & \hat{R} + \frac{1}{h_M} \left[\hat{e}_3, \hat{e}_1 - \hat{e}_2, \hat{e}_4, h_M \hat{e}_1 - \hat{e}_3 \right] \hat{N} \\ & \left[\hat{e}_3, \hat{e}_1 - \hat{e}_2, \hat{e}_4, h_M \hat{e}_1 - \hat{e}_3 \right]^T > 0, \end{aligned} \quad (3.1.45)$$

where

$$\hat{\Gamma}_{[h(t)]} = \begin{bmatrix} \hat{\Gamma} \\ h(t)\hat{e}_{10}^T - \hat{e}_6^T \\ (h_M - h(t))\hat{e}_{11}^T - \hat{e}_7^T \end{bmatrix},$$

where the notations are defined as (3.1.40).

Proof Let us see the following LKFs:

$$\hat{V}(t) = \hat{V}_1 + \hat{V}_2 + \hat{V}_3 + \hat{V}_4, \quad (3.1.46)$$

where

$$\begin{aligned} \hat{V}_1 &= \hat{\zeta}_1^T(t, s) \hat{R} \hat{\zeta}_1(t, s), \\ \hat{V}_2 &= \int_{t-h_M}^t \hat{\zeta}_2^T(t, s) \hat{N} \hat{\zeta}_2(t, s) ds \\ &+ 2 \sum_{i=1}^n \left(\hat{\lambda}_{1,i} \int_0^{W_{2,i}x(t)} (f_i(s) - k_i^- s) ds + \hat{\delta}_{1,i} \int_0^{W_{2,i}x(t)} (k_i^+ s - f_i(s)) ds \right) \\ &+ 2 \sum_{i=1}^n \left(\hat{\lambda}_{2,i} \int_0^{W_{2,i}x(t-h_M)} (f_i(s) - k_i^- s) ds \right. \\ &\left. + \hat{\delta}_{2,i} \int_0^{W_{2,i}x(t-h_M)} (k_i^+ s - f_i(s)) ds \right), \\ \hat{V}_3 &= \int_{t-h(t)}^t \hat{\zeta}_3^T(t, s) \hat{G} \hat{\zeta}_3(t, s) ds, \\ \hat{V}_4 &= \int_{t-h_M}^t \int_s^t \hat{\beta}^T(u) \hat{Q} \hat{\beta}(u) du ds. \end{aligned}$$

Table 3.1.1: Maximum bounds of NNTVDs with different h_D (Example 3.1.1).

h_D	0.8	0.9	unknown
M.J Park <i>et al.</i> [11]	6.7001	4.0747	3.6066
Z. Feng <i>et al.</i> [24]	6.7186	3.9623	3.6654
B. Yang <i>et al.</i> [26]	7.1718	4.2979	3.7615
J.A Wang <i>et al.</i> [25]	7.2033	4.0663	3.6876
O.M Kwon <i>et al.</i> [12]	9.3998	4.6035	3.8840
Ours (Theorem 3.1.1)	6.8886	4.2175	3.6353
Ours (Theorem 3.1.2)	7.0061	4.2954	3.7183
Ours (Theorem 3.1.3)	9.3612	5.1996	4.2019
Ours (Corollary 3.1.3)	7.8554	4.7847	4.2018

With the similar processes of Theorem 3.1.1-3.1.3, one can easily obtain the following condition

$$\begin{aligned}
\dot{\hat{V}}(t) &\leq \hat{\zeta}^T(t) \left(\hat{\Xi}_{[h(t)]} + h(t) \left\{ \hat{S}_1(\hat{Q}_1^{aug})^{-1} \hat{S}_1^T + \frac{1}{3} \hat{U}_1(\hat{Q}_1^{aug})^{-1} \hat{U}_1^T \right\} \right. \\
&\quad \left. + (h_M - h(t)) \left\{ \hat{S}_2(\hat{Q}_2^{aug})^{-1} \hat{S}_2^T + \frac{1}{3} \hat{U}_2(\hat{Q}_2^{aug})^{-1} \hat{U}_2^T \right\} \right) \hat{\zeta}(t) < 0, \\
&\text{subject to } 0 = \hat{\Gamma}_{[h(t)]} \hat{\zeta}(t)
\end{aligned} \tag{3.1.47}$$

and

$$\hat{V}(t) > \hat{\zeta}_1^T(t, s) \hat{R} \hat{\zeta}_1(t, s) + \int_{t-h_M}^t \hat{\zeta}_2^T(t, s) \hat{N} \hat{\zeta}_2(t, s) ds. \tag{3.1.48}$$

The other process is similar to the proofs of Theorems 3.1.1-3.1.3, so it is omitted.

■

Table 3.1.2: Maximum bounds of NNTVDs with different h_D (Example 3.1.2).

h_D	0.40	0.45	0.50	0.55
M.J Park <i>et al.</i> [11]	17.2697	12.0698	10.2903	9.3879
Z. Feng <i>et al.</i> [24]	18.8262	12.8039	10.6230	9.5038
B. Yang <i>et al.</i> [26]	20.8249	14.0082	11.5834	10.2690
J.A Wang <i>et al.</i> [25]	21.0344	14.4707	11.9667	10.6439
O.M Kwon <i>et al.</i> [12]	27.7002	19.5050	16.4611	14.9900
Ours (Theorem 3.1.1)	17.2194	12.2222	10.5582	9.7551
Ours (Theorem 3.1.2)	17.5781	12.5651	10.8613	10.0089
Ours (Theorem 3.1.3)	32.3833	22.5316	19.5912	18.3621
Ours (Corollary 3.1.3)	31.4262	21.2865	17.7803	16.1689

3.1.3 Numerical Examples

Three examples show that the proposed methods can be effective and provide less conservative results in this paper.

Example 3.1.1 Let us consider the system (3.1.1):

$$\begin{aligned}
 A &= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad W_0 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \quad W_1 = \begin{bmatrix} 0.88 & 1 \\ 1 & 1 \end{bmatrix}, \\
 W_2 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad K_p = \text{diag}\{0.4, 0.8\}, \quad K_m = 0_2.
 \end{aligned} \tag{3.1.49}$$

In Table 3.1.2, the maximum delay bounds are shown for different values h_D . Table 3.1.2 includes some recent results such as [11], [12], [24], and [25]. By comparing the results, it is explicitly verified that the proposed methods are enhanced. When $h_D = 0.8$, the upper bound value is smaller than the result in [12]. Through the comparison of Theorem 3.1.3 and Corollary 3.1.3, the proposed LKFs are more

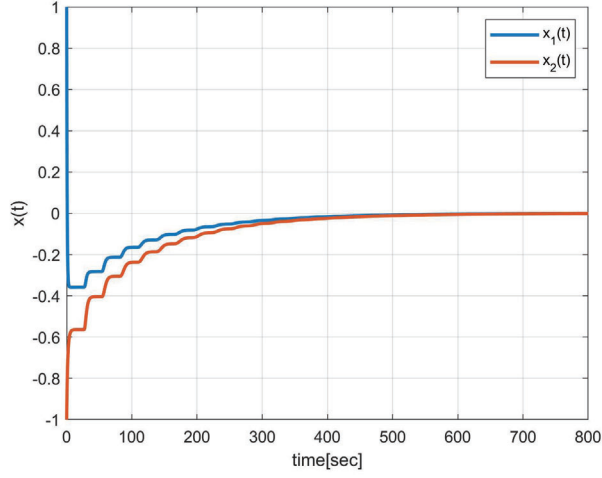


Figure 3.1.1: State trajectories of system (3.1.1) with $h(t) = 0.3 \sin^2(t) + 32.0833$ (Example 3.1.2).

effective with less conservatism. It can have a meaning of the comparisons since others of h_D are larger. Lastly, Theorem 3.1.3 increases the maximum delay bounds and decreases the free variables $\Psi_i \in \mathbb{R}^{n \times 18n} (i = 1, 2)$ by utilizing the AZEA compared to Theorem 3.1.2.

Example 3.1.2 Let us consider the system (3.1.1):

$$A = \begin{bmatrix} 1.5 & 0 \\ 0 & 0.7 \end{bmatrix}, \quad W_0 = \begin{bmatrix} 0.0503 & 0.0454 \\ 0.0987 & 0.2075 \end{bmatrix}, \quad W_1 = \begin{bmatrix} 0.2381 & 0.9320 \\ 0.0388 & 0.5062 \end{bmatrix},$$

$$W_2 = I_2, \quad K_p = \text{diag}\{0.3, 0.8\}, \quad K_m = 0_2. \quad (3.1.50)$$

When the values of h_D are 0.40, 0.45, 0.50, and 0.55, the maximum delay bounds are recorded in Table 3.1.1. It is explicitly verified that the proposed methods are enhanced than the previous literatures [11], [12], [24], and [25]. To check the effectiveness of the results for $h_D = 0.40$ and $h_M = 32.3833$, the state trajectories of Example 3.1.2 with time-varying delay $h(t) = 0.3 \sin^2(t) + 32.0833$, activation function

Table 3.1.3: Maximum bounds of NNTVDs with different h_D (Example 3.1.3).

h_D	0.1	0.5	0.9
M.J Park <i>et al.</i> [11]	1.1194	0.4599	0.4027
B. Yang <i>et al.</i> [26]	1.1205	0.4614	0.3963
Z. Feng <i>et al.</i> [24]	1.1365	0.4678	-
J.A Wang <i>et al.</i> [25]	1.1366	0.4887	0.4324
O.M Kwon <i>et al.</i> [12]	1.1705	0.5437	0.4620
Ours (Theorem 3.1.1)	1.1201	0.4614	0.4031
Ours (Theorem 3.1.2)	1.1202	0.4614	0.4032
Ours (Theorem 3.1.3)	1.1814	0.6687	0.5938
Ours (Corollary 3.1.3)	1.1758	0.6668	0.5938

$f(x(t)) = [0.3 \tanh x_1(t) \ 0.8 \tanh x_2(t)]^T$ and initial condition $x(0) = [1 \ -1]^T$ are shown in Fig. 1. It is proved that the system (3.1.1) with system parameters (3.1.1) is asymptotically stable since the state trajectories of Fig. 3.1.1 converge to zero as time goes to infinity.

Example 3.1.3 Let us consider the system (3.1.1):

$$\begin{aligned}
 A &= \text{diag}\{7.3458, 6.9987, 5.5949\}, \quad W_0 = 0_3, \quad W_1 = I_3, \\
 W_2 &= \begin{bmatrix} 13.6014 & -2.9616 & -0.6936 \\ 7.4736 & 21.6810 & 3.2100 \\ 0.7290 & -2.6334 & -20.1300 \end{bmatrix}, \\
 K_p &= \text{diag}\{0.3680, 0.1795, 0.2876\}, \quad K_m = 0_3.
 \end{aligned} \tag{3.1.51}$$

According to different h_D , it is confirmed that the maximum delay bounds are listed in Table 3.1.3. It includes some recent results [11], [12], [24], [25], and [26]. By comparing the results in Table 3.1.3, the proposed results guarantee less conservative

regions than existing literatures. It is verified that the AZEA is effective technique for increasing the upper bounds delay.

3.1.4 Conclusion

In this paper, enhanced delay-dependent stability criteria for Generalized NNTVDs system are proposed. By constructing the new augmented LKFs considering the activation function-based terms and utilizing GFWMI, the validity guaranteeing asymptotic stability for NNTVDs was illustrated in Theorem 3.1.1. Based on Theorem 3.1.1, the AEZA in [35] is applied to a more relaxed criterion, eliminating free-variables caused zero equality and expanding the maximum delay bounds. Three numerical examples are used to verify the validity and superiority than previous literature. After the predicting the maximum delay bounds, the results are checked by the graph as shown in Figure 3.1.1.

3.2 Reachable set estimation of delayed Markovian jump neural networks

In this section, the estimation of the reachable set in Markovian jump NNTVDs is considered. This section's aims are to estimate an appropriate reachable set for the delayed Markovian jump NNTVDs based on the LKF approach and to apply it in a variety of fields such as conflict resolution between two aircraft [41], a robotic arm model with two joints [42], and an adaptive cruise control system [43]. To utilize the Lyapunov method, the improved LKFs are firstly constructed to estimate allowable smaller reachable sets. Especially in solving the LKFs, this technique is rarely used in other studies and helps to improve the results. And then, a generalized free-weighting-matrix inequality is utilized when the inequality of the LKFs is derived. Some mathematical techniques, such as zero equalities and AZEA, can obtain a significant effect like the improved results in section 3.1.

With the new LKFs proposed this section, LMIs conditions of a delay-dependent are derived for Markovian jump NNTVDs in Theorem 3.2.1. Likewise, using an augmented zero equality approach, based on the extended version of Finsler's lemma, the following notions are proposed in Theorem 3.2.2. With the principles of Theorems 3.2.1-3.2.2, two numerical examples prove the effectiveness and novelty of the proposed results more than previous researches.

3.2.1 Problem Statements

Let us consider the following system for Markovian jump NNTVDs

$$\begin{aligned}\dot{x}(t) &= -A(r_t)x(t) + W_0(r_t)f(W_1x(t)) + W_2(r_t)f(W_1x(t-h(t))) + B(r_t)\omega(t), \quad \forall t > 0, \\ x(t) &= \phi(t), t \in [-h, 0],\end{aligned}\tag{3.2.1}$$

where $x(t) \in \mathbb{R}^n$ is the neuron state vector, the initial condition $\phi(t)$ is a continuously differentiable function. The activation function is defined as $f(\cdot) \in \mathbb{R}^n =$

$[f_1(\cdot), f_2(\cdot), \dots, f_n(\cdot)]^T$. $h(t)$ means the time-varying delays satisfying

$$0 \leq h(t) \leq h_M, \quad \dot{h}(t) \leq h_D, \quad (3.2.2)$$

where h_M and h_D are constants. $\omega^T(t) \in \mathbb{R}^l$ means external input with the form of bounded peak

$$\omega^T(t)\omega(t) \leq \omega_M \quad (3.2.3)$$

where ω_M is a constant. r_t is a right-continuous Markov chain on the probability space having the values in a finite-state space $N = \{1, 2, \dots, m\}$ with generator $\Pi = \{\pi_{pq}\}$, which is represented by

$$P\{r_{t+\Delta t} = q | r_t = p\} = \begin{cases} \pi_{pq}\Delta t + o(\Delta t), & p \neq q \\ 1 + \pi_{pp}\Delta t + o(\Delta t), & p = q \end{cases} \quad (3.2.4)$$

where $\Delta t > 0$ and $\lim_{\Delta t \rightarrow 0} \frac{o(\Delta t)}{\Delta t} = 0$. If $p \neq q$, $\pi_{pq} \geq 0$ is the transition rate from mode p at time t to mode q at time $t + \Delta t$, and $\pi_{pp} = -\sum_{q=1, q \neq p}^m \pi_{pq}$. $A(r_t) \in \mathbb{R}^{n \times n}$ is a positive diagonal matrix, $W_0(r_t)$, W_1 , and $W_2(r_t) \in \mathbb{R}^{n \times n}$ are the interconnection matrices between neurons.

This paper aims to estimate a reachable set for Markovian jump NNTVDs (3.2.1) with time-varying delays. By constructing LKFs and utilizing the following definition and some mathematical techniques, enhanced stability conditions will be derived in the forms of LMIs.

Definition 1. For $\forall t \geq 0$, the reachable set of system (3.2.1) is denoted as follows:

$$R_x = \{x(t) | x(t) \text{ and } \omega(t) \text{ satisfy (1) with (2) and (3)}\} \quad (3.2.5)$$

and an ellipsoid used to bound the reachable set of Markovian jump NNTVDs (3.2.1) is given in the form of

$$\epsilon(P) = \{\xi | \xi^T P \xi \leq 1, \xi \in \mathbb{R}^n, P > 0\}.$$

3.2.2 Main Results

This paper shows the main results with the Markovian jump NNTVDs and some Lemmas. To simply the expression of the main results, block entry matrices are defined to represent matrices and vector states such that $e_i (i = 1, 2, \dots, 19) \in \mathbb{R}^{(15n-l) \times n}$. For examples, $e_3 = [0_{n \times 2n}, I_n, 0_{n \times 16n}]^T$ and $e_{15} = [0_{n \times 14n}, I_n, 0_{n \times 4n}]^T$.

The other notations are defined as

$$\begin{aligned}
K_p &= \text{diag}\{k_1^+, \dots, k_n^+\}, \quad K_m = \text{diag}\{k_1^-, \dots, k_n^-\}, \\
\zeta(t) &= \left[x^T(t), x^T(t-h(t)), x^T(t-h_M), \dot{x}^T(t), \dot{x}^T(t-h_M), \frac{1}{h(t)} \int_{t-h(t)}^t x^T(s) ds, \right. \\
&\quad \frac{1}{h_M-h(t)} \int_{t-h_M}^{t-h(t)} x^T(s) ds, f^T(W_1 x(t)), f^T(W_1 x(t-h(t))), \\
&\quad f^T(W_1 x(t-h_M)), \frac{1}{h(t)} \int_{t-h(t)}^t \int_s^t x^T(u) du ds, \\
&\quad \frac{1}{h_M-h(t)} \int_{t-h_M}^{t-h(t)} \int_s^{t-h(t)} x^T(u) du ds, \frac{1}{h(t)} \int_{t-h(t)}^t \int_s^t f^T(W_1 x(u)) du ds, \\
&\quad \frac{1}{h_M-h(t)} \int_{t-h_M}^{t-h(t)} \int_s^{t-h(t)} f^T(W_1 x(u)) du ds, \\
&\quad \left. \int_{t-h(t)}^t x^T(s) ds, \int_{t-h_M}^{t-h(t)} x^T(s) ds, \omega^T(t) \right]^T, \\
\eta_1(t) &= \left[x^T(t), x^T(t-h_M), \int_{t-h_M}^t x^T(s) ds \right]^T, \\
\eta_2(s) &= [x^T(s), \dot{x}^T(s), f^T(W_1 x(s))]^T, \\
R_{aug1} &= R - \alpha e^{\alpha h_M} \begin{bmatrix} Q_{2,11} & 0_n & Q_{2,12} \\ 0_n & 0_n & 0_n \\ Q_{2,21} & 0_n & Q_{2,22} \end{bmatrix} + \begin{bmatrix} 0_n & P_1 & 0_n \\ \star & 0_n & 0_n \\ \star & \star & 0_n \end{bmatrix}, \\
R_{aug2} &= R + \begin{bmatrix} 0_n & P_2 & 0_n \\ \star & 0_n & 0_n \\ \star & \star & 0_n \end{bmatrix}, \\
\Xi_1^m &= \alpha \left(e_1 U_m e_1^T + [e_1, e_3, e_{17} + e_{18}] P [e_1, e_3, e_{17} + e_{18}]^T \right)
\end{aligned}$$

$$\begin{aligned}
& +Sym \left\{ e_1 U_m e_4^T + [e_1, e_3, e_{17} + e_{18}] P [e_4, e_5, e_1 - e_3]^T \right\} + \sum_{p=1}^N \pi_{mp} e_1 U_m e_1^T, \\
\Xi_2 &= [e_1, e_4, e_8] Q_1 [e_1, e_4, e_8]^T - e^{-\alpha h_M} [e_3, e_5, e_{10}] Q_1 [e_3, e_5, e_{10}]^T \\
& +Sym \left\{ [e_8 - e_1 W_1^T K_m^T] \Lambda_1 W_1 e_4^T + [e_1 W_1^T K_p^T - e_8] \Delta_1 W_1 e_4^T \right\} \\
& +Sym \left\{ [e_{10} - e_3 W_1^T K_m^T] \Lambda_2 W_1 e_5^T + [e_3 W_1^T K_p^T - e_{10}] \Delta_2 W_1 e_5^T \right\}, \\
\Xi_3 &= [e_1, e_8] Q_2 [e_1, e_8]^T - (1 - \mu) [e_2, e_9] Q_2 [e_2, e_9]^T, \\
\Xi_4 &= h_M [e_1, e_4, e_8] R [e_1, e_4, e_8]^T, \\
\Xi_{41} &= e^{-\alpha h_M} Sym \left\{ S_1 [e_{17}, e_1 - e_2, e_{11}]^T \right. \\
& \quad \left. + N_1 [-e_{17} + 2e_{13}, e_1 + e_2 - 2e_6, -e_{11} + 2e_{15}]^T \right\}, \\
\Xi_{42} &= e^{-\alpha h_M} Sym \left\{ S_2 [e_{18}, e_2 - e_3, e_{12}]^T \right. \\
& \quad \left. + N_2 [-e_{18} + 2e_{14}, e_2 + e_3 - 2e_7, -e_{12} + 2e_{16}]^T \right\}, \\
\Xi_5 &= e^{-\alpha h_M} (e_1 P_1 e_1^T - e_2 (P_1 - P_2) e_2^T - e_3 P_2 e_3^T), \\
\Xi_{6[h(t)]} &= Sym \{ (h(t) e_6 - e_{17}) \Psi_1 \} + Sym \{ ((h_M - h(t)) e_7 - e_{18}) \Psi_2 \}, \\
\Theta_{1[h(t)]} &= -Sym \left\{ (e_8 - e_1 W_1^T K_m) \left(\frac{h(t)}{h_M} H_1 + \frac{h_M - h(t)}{h_M} F_1 \right) (e_8 - e_1 W_1^T K_p)^T \right\} \\
& -Sym \left\{ (e_9 - e_2 W_1^T K_m) \left(\frac{h(t)}{h_M} H_2 + \frac{h_M - h(t)}{h_M} F_2 \right) (e_9 - e_2 W_1^T K_p)^T \right\} \\
& -Sym \left\{ (e_{10} - e_3 W_1^T K_m) \left(\frac{h(t)}{h_M} H_3 + \frac{h_M - h(t)}{h_M} F_3 \right) \right. \\
& \quad \left. \times (e_{10} - e_3 W_1^T K_p)^T \right\}, \\
\Theta_{2[h(t)]} &= -Sym \left\{ (e_8 - e_9 - (e_1 - e_2) W_1^T K_m) \left(\frac{h(t)}{h_M} H_4 + \frac{h_M - h(t)}{h_M} F_4 \right) \right. \\
& \quad \left. \times (e_8 - e_9 - (e_1 - e_2) W_1^T K_p)^T \right\} \\
& -Sym \left\{ (e_8 - e_{10} - (e_1 - e_3) W_1^T K_m) \left(\frac{h(t)}{h_M} H_5 + \frac{h_M - h(t)}{h_M} F_5 \right) \right. \\
& \quad \left. \times (e_8 - e_{10} - (e_1 - e_3) W_1^T K_p)^T \right\} \\
& -Sym \left\{ (e_9 - e_{10} - (e_2 - e_3) W_1^T K_m) \left(\frac{h(t)}{h_M} H_6 + \frac{h_M - h(t)}{h_M} F_6 \right) \right. \\
& \quad \left. \times (e_9 - e_{10} - (e_2 - e_3) W_1^T K_p)^T \right\},
\end{aligned}$$

$$\Xi_{[h(t)]} = \Xi_1 + \Xi_2 + \Xi_3 + \Xi_4 + \Xi_{41} + \Xi_{42} + \Xi_5 + \Xi_{6[h(t)]} + \Theta_{1[h(t)]} + \Theta_{2[h(t)]},$$

$$\Gamma_m = -A_m e_1^T - e_4^T + W_{0m} e_8^T + W_{2m} e_9^T + B e_{19}^T. \quad (3.2.6)$$

Then, the following theorem is given by main result.

Theorem 3.2.1 *For given scalars h_M and μ , system (3.2.1) is asymptotically stable for $0 \leq h(t) \leq h_M$ and $\dot{h}(t) \leq \mu$, if there exist scalar $\alpha > 0$, matrices $U_m \in \mathbb{S}_+^n$, $P, R, Q_1 \in \mathbb{S}_+^{3n}$, $Q_2 \in \mathbb{S}_+^{2n}$, $\Lambda_i, \Delta_i \in \mathbb{D}_+^n$, $P_i \in \mathbb{S}^n$, $S_i, N_i \in \mathbb{R}^{(19n-l) \times 3n}$, $\Psi_i \in \mathbb{R}^{n \times (19n-l)}$ ($i = 1, 2$), $H_j, F_j \in \mathbb{D}_+^n$ ($j = 1, 2, \dots, 6$) satisfying the following LMIs:*

$$\begin{bmatrix} (\Gamma^\perp)^T (\Xi_{[h_M]}^m - \beta e_{19} e_{19}^T) (\Gamma^\perp) & (\Gamma^\perp)^T (h_M S_1) & (\Gamma^\perp)^T (h_M N_1) \\ \star & -h_M e^{\alpha h_M} R_{aug1} & 0_{3n} \\ \star & \star & -3h_M e^{\alpha h_M} R_{aug1} \end{bmatrix} < 0, \quad (3.2.7)$$

$$\begin{bmatrix} (\Gamma^\perp)^T (\Xi_{[0]}^m - \beta e_{19} e_{19}^T) (\Gamma^\perp) & (\Gamma^\perp)^T (h_M S_2) & (\Gamma^\perp)^T (h_M N_2) \\ \star & -h_M e^{\alpha h_M} R_{aug2} & 0_{3n} \\ \star & \star & -3h_M e^{\alpha h_M} R_{aug2} \end{bmatrix} < 0, \quad (3.2.8)$$

$$R_{aug1} \geq 0, \quad (3.2.9)$$

$$R_{aug2} \geq 0, \quad (3.2.10)$$

where the notations are defined as (3.2.6).

Proof Consider the LKFs candidate:

$$V(x(t), m) = V_1(x(t), m) + \sum_{i=2}^4 V_i(x(t)), \quad (3.2.11)$$

where

$$\begin{aligned}
V_1(x(t), m) &= x^T(t)U_mx(t) + \eta_1^T(t)P\eta_1(t), \\
V_2(x(t)) &= \int_{t-h_M}^t e^{\alpha(s-t)}\eta_2^T(s)Q_1\eta_2(s)ds \\
&\quad + 2e^{-\alpha t} \sum_{i=1}^n \int_0^{W_{1,i}x(t)} [\lambda_{1,i}(f_i(s) - k_i^-s) + \delta_{1,i}(k_i^+s - f_i(s))] ds \\
&\quad + 2e^{-\alpha t} \sum_{i=1}^n \int_0^{W_{1,i}x(t-h_M)} [\lambda_{2,i}(f_i(s) - k_i^-s) + \delta_{2,i}(k_i^+s - f_i(s))] ds, \\
V_3(x(t)) &= \int_{t-h(t)}^t \begin{bmatrix} x(s) \\ f(W_1x(s)) \end{bmatrix}^T Q_2 \begin{bmatrix} x(s) \\ f(W_1x(s)) \end{bmatrix} ds, \\
V_4(x(t)) &= \int_{t-h_M}^t \int_s^t e^{\alpha(u-t)}\eta_2^T(u)R\eta_2(u)duds.
\end{aligned}$$

Let $\mathfrak{L}$ be the weak-infinitesimal generator, and it acts on a $V(x(t), m)$.

$\mathfrak{L}\{V_1(x(t), s)\}$ and $\mathfrak{L}\{V_2(x(t))\}$ are written as

$$\begin{aligned}
\mathfrak{L}\{V_1(x(t), m)\} &= 2x^T(t)U_m\dot{x}(t) + 2\eta_1^T(t)P\dot{\eta}_1(t) + \sum_{m=1}^N \pi_{mp}x^T(t)U_mx(t) \\
&= -\alpha(x^T(t)U_mx(t) + \eta_1^T(t)P\eta_1(t)) + \alpha(x^T(t)U_mx(t) + \eta_1^T(t)P\eta_1(t)) \\
&\quad + 2x^T(t)U_m\dot{x}(t) + 2\eta_1^T(t)P\dot{\eta}_1(t) + \sum_{m=1}^N \pi_{mp}x^T(t)U_mx(t) \\
&= -\alpha V_1(x(t), m) + \zeta^T(t)\Xi_1^m\zeta(t), \tag{3.2.12}
\end{aligned}$$

and

$$\begin{aligned}
\mathfrak{L}\{V_2(x(t))\} &= \begin{bmatrix} x(t) \\ \dot{x}(t) \\ f(W_1x(t)) \end{bmatrix}^T Q_1 \begin{bmatrix} x(t) \\ \dot{x}(t) \\ f(W_1x(t)) \end{bmatrix} \\
&\quad - e^{-\alpha h_M} \begin{bmatrix} x(t-h_M) \\ \dot{x}(t-h_M) \\ f(W_1x(t-h_M)) \end{bmatrix}^T Q_1 \begin{bmatrix} x(t-h_M) \\ \dot{x}(t-h_M) \\ f(W_1x(t-h_M)) \end{bmatrix} \\
&\quad + \int_{t-h_M}^t \frac{d}{dt} \left(e^{\alpha(s-t)}\eta_2^T(s)Q_1\eta_2(s) \right) ds
\end{aligned}$$

$$\begin{aligned}
& -2\alpha e^{-\alpha t} \sum_{i=1}^n \int_0^{W_{1,i}x(t)} [\lambda_{1,i} (f_i(s) - k_i^- s) + \delta_{1,i} (k_i^+ s - f_i(s))] ds \\
& -2\alpha e^{-\alpha t} \sum_{i=1}^n \int_0^{W_{1,i}x(t-h_M)} [\lambda_{2,i} (f_i(s) - k_i^- s) + \delta_{2,i} (k_i^+ s - f_i(s))] ds \\
& + 2e^{-\alpha t} [f(W_1x(t)) - K_m W_1x(t)]^T \Lambda_1 W_1 \dot{x}(t) \\
& + 2e^{-\alpha t} [K_p W_1x(t) - f(W_1x(t))]^T \Delta_1 W_1 \dot{x}(t) \\
& + 2e^{-\alpha t} [f(W_1x(t-h_M)) - K_m W_1x(t-h_M)]^T \Lambda_2 W_1 \dot{x}(t-h_M) \\
& + 2e^{-\alpha t} [K_p W_1x(t-h_M) - f(W_1x(t-h_M))]^T \Delta_2 W_1 \dot{x}(t-h_M) \\
& \leq -\alpha V_2(x(t)) + \zeta^T(t) \Xi_2 \zeta(t).
\end{aligned} \tag{3.2.13}$$

The upper bounds of $\mathfrak{L}\{V_3(x(t))\}$ can lead to

$$\begin{aligned}
\mathfrak{L}\{V_3(x(t))\} &= \begin{bmatrix} x(t) \\ f(W_1x(t)) \end{bmatrix}^T Q_2 \begin{bmatrix} x(t) \\ f(W_1x(t)) \end{bmatrix} \\
&\quad - (1 - \dot{h}(t)) \begin{bmatrix} x(t-h(t)) \\ f(W_1x(t-h(t))) \end{bmatrix}^T Q_2 \begin{bmatrix} x(t-h(t)) \\ f(W_1x(t-h(t))) \end{bmatrix} \\
&\leq -\alpha V_3(x(t)) + \zeta^T(t) \Xi_3 \zeta(t) \\
&\quad + \alpha \int_{t-h(t)}^t \begin{bmatrix} x(s) \\ f(W_1x(s)) \end{bmatrix}^T Q_2 \begin{bmatrix} x(s) \\ f(W_1x(s)) \end{bmatrix} ds.
\end{aligned} \tag{3.2.14}$$

Calculating $\mathfrak{L}\{V_4(x(t))\}$ is represented

$$\begin{aligned}
\mathfrak{L}\{V_4(x(t))\} &= -\alpha \int_{t-h_M}^t \int_s^t e^{\alpha(u-t)} \begin{bmatrix} x(u) \\ \dot{x}(u) \\ f(W_1x(u)) \end{bmatrix}^T R \begin{bmatrix} x(u) \\ \dot{x}(u) \\ f(W_1x(u)) \end{bmatrix} du ds \\
&\quad - \int_{t-h_M}^t e^{\alpha(s-t)} \begin{bmatrix} x(s) \\ \dot{x}(s) \\ f(W_1x(s)) \end{bmatrix}^T R \begin{bmatrix} x(s) \\ \dot{x}(s) \\ f(W_1x(s)) \end{bmatrix} ds
\end{aligned}$$

$$\begin{aligned}
& +h_M \begin{bmatrix} x(t) \\ \dot{x}(t) \\ f(W_1x(t)) \end{bmatrix}^T R \begin{bmatrix} x(t) \\ \dot{x}(t) \\ f(W_1x(t)) \end{bmatrix} \\
& = -\alpha V_4(x(t)) + h_M \begin{bmatrix} x(t) \\ \dot{x}(t) \\ f(W_1x(t)) \end{bmatrix}^T R \begin{bmatrix} x(t) \\ \dot{x}(t) \\ f(W_1x(t)) \end{bmatrix} \\
& \quad - \int_{t-h_M}^t e^{\alpha(s-t)} \begin{bmatrix} x(s) \\ \dot{x}(s) \\ f(W_1x(s)) \end{bmatrix}^T R \begin{bmatrix} x(s) \\ \dot{x}(s) \\ f(W_1x(s)) \end{bmatrix} ds. \quad (3.2.15)
\end{aligned}$$

Considering that $t - h \leq s \leq t$, one can easily obtain that $e^{-\alpha h} \leq e^{\alpha(s-t)} \leq 1$,

$$\begin{aligned}
& \mathfrak{L}\{V_4(x(t))\} + \alpha \int_{t-h(t)}^t \begin{bmatrix} x(s) \\ f(W_1x(s)) \end{bmatrix}^T Q_2 \begin{bmatrix} x(s) \\ f(W_1x(s)) \end{bmatrix} ds \\
& \leq -\alpha V_4(x(t)) + h_M \begin{bmatrix} x(t) \\ \dot{x}(t) \\ f(W_1x(t)) \end{bmatrix}^T R \begin{bmatrix} x(t) \\ \dot{x}(t) \\ f(W_1x(t)) \end{bmatrix} \\
& \quad - e^{-\alpha h} \int_{t-h_M}^t \begin{bmatrix} x(s) \\ \dot{x}(s) \\ f(W_1x(s)) \end{bmatrix}^T R \begin{bmatrix} x(s) \\ \dot{x}(s) \\ f(W_1x(s)) \end{bmatrix} ds \\
& = -\alpha V_4(x(t)) + \zeta^T(t) \Xi_4 \zeta(t) \\
& \quad - e^{-\alpha h} \int_{t-h_M}^t \begin{bmatrix} x(s) \\ \dot{x}(s) \\ f(W_1x(s)) \end{bmatrix}^T R \begin{bmatrix} x(s) \\ \dot{x}(s) \\ f(W_1x(s)) \end{bmatrix} ds \\
& \quad + \alpha \int_{t-h(t)}^t \begin{bmatrix} x(s) \\ f(W_1x(s)) \end{bmatrix}^T Q_2 \begin{bmatrix} x(s) \\ f(W_1x(s)) \end{bmatrix} ds. \quad (3.2.16)
\end{aligned}$$

The following two zero equalities are considered as follows:

$$0 = e^{-\alpha h} \left(x^T(t) P_1 x(t) - x^T(t-h(t)) P_1 x(t-h(t)) - 2 \int_{t-h(t)}^t x^T(s) P_1 \dot{x}(s) ds \right), \quad (3.2.17)$$

$$0 = e^{-\alpha h} \left(x^T(t-h(t)) P_2 x(t-h(t)) - x^T(t-h_M) P_2 x(t-h_M) - 2 \int_{t-h_M}^{t-h(t)} x^T(s) P_2 \dot{x}(s) ds \right). \quad (3.2.18)$$

The sum of (3.2.17) and (3.2.18) can lead to

$$0 = \zeta^T(t) \Xi_5 \zeta(t) - 2e^{-\alpha h} \int_{t-h(t)}^t x^T(s) P_1 \dot{x}(s) ds - 2e^{-\alpha h} \int_{t-h_M}^{t-h(t)} x^T(s) P_2 \dot{x}(s) ds. \quad (3.2.19)$$

By adding (3.2.19) into (3.2.16), it can be

$$\begin{aligned} & \mathfrak{L} \{V_4(x(t))\} + \alpha \int_{t-h(t)}^t \begin{bmatrix} x(s) \\ f(W_1 x(s)) \end{bmatrix}^T Q_2 \begin{bmatrix} x(s) \\ f(W_1 x(s)) \end{bmatrix} ds \\ & \leq \zeta^T(t) (\Xi_4 + \Xi_5) \zeta(t) - e^{-\alpha h_M} \int_{t-h(t)}^t \begin{bmatrix} x(s) \\ \dot{x}(s) \\ f(W_1 x(s)) \end{bmatrix}^T \\ & \quad \times \left(R - \alpha e^{\alpha h_M} \begin{bmatrix} Q_{2,11} & 0_n & Q_{2,12} \\ 0_n & 0_n & 0_n \\ Q_{2,21} & 0_n & Q_{2,22} \end{bmatrix} + \begin{bmatrix} 0_n & P_1 & 0_n \\ \star & 0_n & 0_n \\ \star & \star & 0_n \end{bmatrix} \right) \\ & \quad \times \begin{bmatrix} x(s) \\ \dot{x}(s) \\ f(W_1 x(s)) \end{bmatrix} ds \end{aligned}$$

$$\begin{aligned}
& - \int_{t-h_M}^{t-h(t)} \begin{bmatrix} x(s) \\ \dot{x}(s) \\ f(W_1 x(s)) \end{bmatrix}^T \left(R + \begin{bmatrix} 0_n & P_2 & 0_n \\ \star & 0_n & 0_n \\ \star & \star & 0_n \end{bmatrix} \right) \begin{bmatrix} x(s) \\ \dot{x}(s) \\ f(W_1 x(s)) \end{bmatrix} ds \\
& = \zeta^T(t) (\Xi_4 + \Xi_5) \zeta(t) \\
& - e^{-\alpha h_M} \int_{t-h(t)}^t \begin{bmatrix} x(s) \\ \dot{x}(s) \\ f(W_1 x(s)) \end{bmatrix}^T R_{aug1} \begin{bmatrix} x(s) \\ \dot{x}(s) \\ f(W_1 x(s)) \end{bmatrix} ds \\
& - e^{-\alpha h_M} \int_{t-h_M}^{t-h(t)} \begin{bmatrix} x(s) \\ \dot{x}(s) \\ f(W_1 x(s)) \end{bmatrix}^T R_{aug2} \begin{bmatrix} x(s) \\ \dot{x}(s) \\ f(W_1 x(s)) \end{bmatrix} ds, \tag{3.2.20}
\end{aligned}$$

where R_{aug1} and R_{aug2} are defined as (3.2.6).

From Lemma 2.1.2, the integral terms in (3.2.20) get the upper bounds as

$$\begin{aligned}
& - e^{-\alpha h_M} \int_{t-h(t)}^t \begin{bmatrix} x(s) \\ \dot{x}(s) \\ f(W_1 x(s)) \end{bmatrix}^T R_{aug1} \begin{bmatrix} x(s) \\ \dot{x}(s) \\ f(W_1 x(s)) \end{bmatrix} ds \\
& \leq e^{-\alpha h_M} \text{Sym} \{ \zeta^T(t) S_1 \omega_1(t) + \zeta^T(t) U_1 \omega_2(t) \} + h(t) \zeta^T(t) (S_1 (R_{aug1})^{-1} S_1^T \\
& \quad + \frac{1}{3} U_1 (R_{aug1})^{-1} U_1^T) \zeta(t) \\
& = e^{-\alpha h_M} \zeta^T(t) \Xi_{41} \zeta(t), \tag{3.2.21}
\end{aligned}$$

$$\begin{aligned}
& - e^{-\alpha h_M} \int_{t-h_M}^{t-h(t)} \begin{bmatrix} x(s) \\ \dot{x}(s) \\ f(W_1 x(s)) \end{bmatrix}^T R_{aug2} \begin{bmatrix} x(s) \\ \dot{x}(s) \\ f(W_1 x(s)) \end{bmatrix} ds \\
& \leq e^{-\alpha h_M} \text{Sym} \{ \zeta^T(t) S_2 \omega_3(t) + \zeta^T(t) U_2 \omega_4(t) \} + (h_M - h(t)) \zeta^T(t) (S_2 (R_{aug2})^{-1} S_2^T \\
& \quad + \frac{1}{3} U_2 (R_{aug2})^{-1} U_2^T) \zeta(t) \\
& = e^{-\alpha h_M} \zeta^T(t) \Xi_{42} \zeta(t), \tag{3.2.22}
\end{aligned}$$

where

$$\begin{aligned}
\omega_1(t) &= \int_{t-h(t)}^t \begin{bmatrix} x(s) \\ \dot{x}(s) \\ f(W_1 x(s)) \end{bmatrix} ds = [e_{17}, e_1 - e_2, e_{11}]^T \zeta(t), \\
\omega_2(t) &= - \int_{t-h(t)}^t \begin{bmatrix} x(s) \\ \dot{x}(s) \\ f(W_1 x(s)) \end{bmatrix} ds + \frac{2}{h(t)} \int_{t-h(t)}^t \int_s^t \begin{bmatrix} x(u) \\ \dot{x}(u) \\ f(W_1 x(u)) \end{bmatrix} dud s \\
&= [-e_{17} + 2e_{13}, e_1 + e_2 - 2e_6, -e_{11} + 2e_{15}]^T \zeta(t), \\
\omega_3(t) &= \int_{t-h_M}^{t-h(t)} \begin{bmatrix} x(s) \\ \dot{x}(s) \\ f(W_1 x(s)) \end{bmatrix} ds = [e_{18}, e_2 - e_3, e_{12}]^T \zeta(t), \\
\omega_4(t) &= - \int_{t-h_M}^{t-h(t)} \begin{bmatrix} x(s) \\ \dot{x}(s) \\ f(W_1 x(s)) \end{bmatrix} ds \\
&\quad + \frac{2}{h_M - h(t)} \int_{t-h_M}^{t-h(t)} \int_s^{t-h(t)} \begin{bmatrix} x(u) \\ \dot{x}(u) \\ f(W_1 x(u)) \end{bmatrix} dud s \\
&= [-e_{18} + 2e_{14}, e_2 + e_3 - 2e_7, -e_{12} + 2e_{16}]^T \zeta(t).
\end{aligned}$$

The upper bound of $\mathfrak{L}\{V_4(x(t))\}$ derived using (3.2.15)-(3.2.22) is as follows:

$$\begin{aligned}
&\mathfrak{L}\{V_4(x(t))\} + \alpha \int_{t-h(t)}^t \begin{bmatrix} x(s) \\ f(W_1 x(s)) \end{bmatrix}^T Q_2 \begin{bmatrix} x(s) \\ f(W_1 x(s)) \end{bmatrix} ds \\
&\leq \zeta^T(t) (\Xi_4 + \Xi_{41} + \Xi_{42} + \Xi_5) \zeta(t) \\
&\quad + h(t) \zeta^T(t) \left\{ S_1(R_{aug1})^{-1} S_1^T + \frac{1}{3} U_1(R_{aug1})^{-1} U_1^T \right\} \zeta(t) \\
&\quad + (h_M - h(t)) \zeta^T(t) \left\{ S_2(R_{aug2})^{-1} S_2^T + \frac{1}{3} U_2(R_{aug2})^{-1} U_2^T \right\} \zeta(t). \quad (3.2.23)
\end{aligned}$$

In the component association of $\zeta(t)$, two zero equalities are derived:

$$0 = \zeta^T(t) \text{Sym} \{ (h(t)e_6 - e_{17}) \Psi_1 \} \zeta(t), \quad (3.2.24)$$

$$0 = \zeta^T(t) \text{Sym} \{ ((h_M - h(t)) e_7 - e_{18}) \Psi_2 \} \zeta(t), \quad (3.2.25)$$

where $\Psi_i \in \mathbb{R}^{n \times (19n-l)}$ ($i = 1, 2$) are any matrices.

The sum of (3.2.24) and (3.2.25) can lead to

$$0 = \zeta^T(t) \Xi_{6[h(t)]} \zeta(t). \quad (3.2.26)$$

From Assumption 1 with $a = 0$, for $H_j, F_j \in \mathbb{D}_+^n$ ($j = 1, 2, 3$), the following inequality is satisfied:

$$\zeta^T(t) \Theta_{1[h(t)]} \zeta(t) \geq 0, \quad (3.2.27)$$

where

$$\begin{aligned} \Theta_{1[h(t)]} = & -\text{Sym} \left\{ (e_9 - e_1 W_1^T K_m) \left(\frac{h(t)}{h_M} H_1 + \frac{h_M - h(t)}{h_M} F_1 \right) (e_9 - e_1 W_1^T K_p)^T \right\} \\ & -\text{Sym} \left\{ (e_{10} - e_2 W_1^T K_m) \left(\frac{h(t)}{h_M} H_2 + \frac{h_M - h(t)}{h_M} F_2 \right) (e_{10} - e_2 W_1^T K_p)^T \right\} \\ & -\text{Sym} \left\{ (e_{11} - e_3 W_1^T K_m) \left(\frac{h(t)}{h_M} H_3 + \frac{h_M - h(t)}{h_M} F_3 \right) \right. \\ & \left. \times (e_{11} - e_3 W_1^T K_p)^T \right\}, \end{aligned}$$

and with $a \neq 0$, for $H_j, F_j \in \mathbb{D}_+^n$ ($j = 4, 5, 6$), the following inequality is also satisfied:

$$\zeta^T(t) \Theta_{2[h(t)]} \zeta(t) \geq 0, \quad (3.2.28)$$

where

$$\begin{aligned} \Theta_{2[h(t)]} = & -\text{Sym} \left\{ (e_9 - e_{10} - (e_1 - e_2) W_1^T K_m) \left(\frac{h(t)}{h_M} H_4 + \frac{h_M - h(t)}{h_M} F_4 \right) \right. \\ & \left. \times (e_9 - e_{10} - (e_1 - e_2) W_1^T K_p)^T \right\} \\ & -\text{Sym} \left\{ (e_9 - e_{11} - (e_1 - e_3) W_1^T K_m) \left(\frac{h(t)}{h_M} H_5 + \frac{h_M - h(t)}{h_M} F_5 \right) \right. \\ & \left. \times (e_9 - e_{11} - (e_1 - e_3) W_1^T K_p)^T \right\} \end{aligned}$$

$$\begin{aligned}
& \times (e_9 - e_{11} - (e_1 - e_3) W_1^T K_p)^T \Big\} \\
& - Sym \left\{ (e_{10} - e_{11} - (e_2 - e_3) W_1^T K_m) \left(\frac{h(t)}{h_M} H_6 + \frac{h_M - h(t)}{h_M} F_6 \right) \right. \\
& \left. \times (e_{10} - e_{11} - (e_2 - e_3) W_1^T K_p)^T \right\}.
\end{aligned}$$

The proof of the inequalities (3.2.27) and (3.2.28) was specifically explained in [16].

By using Lemma 2.1.5, an upper bound of $\mathfrak{L}\{V(x(t), m)\}$ (3.2.12)-(3.2.28) can be written as

$$\begin{aligned}
& \mathfrak{L}\{V(x(t), m)\} + \alpha V(x(t), m) - \beta \omega^T(t) \omega(t) \\
& \leq \zeta^T(t) \left[\Xi_{[h(t)]} + \Xi_{6[h(t)]} + h(t) \left\{ S_1(R_{aug1})^{-1} S_1^T + \frac{1}{3} U_1(R_{aug1})^{-1} U_1^T \right\} \right. \\
& \quad + (h_M - h(t)) \left\{ S_2(R_{aug2})^{-1} S_2^T + \frac{1}{3} U_2(R_{aug2})^{-1} U_2^T \right\} \\
& \quad \left. - \beta e_{19} e_{19}^T \right] \zeta(t). \tag{3.2.29}
\end{aligned}$$

A stability condition in the system (3.2.1) can be

$$\begin{aligned}
& \zeta^T(t) \left[\Xi_{[h(t)]} + \Xi_{6[h(t)]} + h(t) \left\{ S_1(R_{aug1})^{-1} S_1^T + \frac{1}{3} U_1(R_{aug1})^{-1} U_1^T \right\} \right. \\
& \quad + (h_M - h(t)) \left\{ S_2(R_{aug2})^{-1} S_2^T + \frac{1}{3} U_2(R_{aug2})^{-1} U_2^T \right\} \\
& \quad \left. - \beta e_{19} e_{19}^T \right] \zeta(t) < 0, \tag{3.2.30}
\end{aligned}$$

subject to $0 = \Gamma \zeta(t)$.

From Lemma 2.1.3, the following inequality satisfies:

$$\begin{aligned}
& \left(\Gamma^\perp \right)^T \left[\Xi_{[h(t)]} + \Xi_{6[h(t)]} + h(t) \left\{ S_1(R_{aug1})^{-1} S_1^T + \frac{1}{3} U_1(R_{aug1})^{-1} U_1^T \right\} \right. \\
& \quad + (h_M - h(t)) \left\{ S_2(R_{aug2})^{-1} S_2^T + \frac{1}{3} U_2(R_{aug2})^{-1} U_2^T \right\} \\
& \quad \left. - \beta e_{19} e_{19}^T \right] \left(\Gamma^\perp \right) < 0. \tag{3.2.31}
\end{aligned}$$

The inequality (3.2.31) is satisfied, if the following two inequalities hold

$$\left(\Gamma^\perp \right)^T \left[\Xi_{[h_M]} + \Xi_{6[h_M]} + h_M \left\{ S_1(R_{aug1})^{-1} S_1^T + \frac{1}{3} U_1(R_{aug1})^{-1} U_1^T \right\} \right.$$

$$-\beta e_{19} e_{19}^T \Big] \left(\Gamma^\perp \right) < 0 \quad (3.2.32)$$

and

$$\begin{aligned} & \left(\Gamma^\perp \right)^T \left[\Xi_{[0]} + \Xi_{6[0]} + h_M \left\{ S_2(R_{aug2})^{-1} S_2^T + \frac{1}{3} U_2(R_{aug2})^{-1} U_2^T \right\} \right. \\ & \left. - \beta e_{19} e_{19}^T \right] \left(\Gamma^\perp \right) < 0. \end{aligned} \quad (3.2.33)$$

since it relies on $h(t)$ with the condition of $0 \leq h(t) \leq h_M$. Therefore, the stability condition for Markovian jump NNTVDs (3.2.1) satisfies with (3.2.7) and (3.2.8). Thus, it is asymptotically stable for $0 \leq h(t) \leq h_M$ and $\dot{h}(t) \leq h_D$. This completes our proof. $\blacksquare$

Remark 8 It was proved that inserting inequalities containing the activation function condition (3.1.3) provides enhanced stability criteria for Markovian jump NNTVDs in [10]. Like [28], $\frac{h(t)}{h_M} H_i + \frac{h_M - h(t)}{h_M} F_i$ ($i = 1, \dots, 5$) are always positive definite since the time-varying delay $0 \leq h(t) \leq h_M$ relies on $h(t)$.

Remark 9 In the previous researches such as [44] and [45], the exponential functions $e^{\alpha(s-t)}$ is included in the every integral terms of the constructing LKFs. In this paper, $\mathfrak{L}\{V_3(x(t))\}$ doesn't contain $-\alpha V_3(x(t))$ since $V_3(x(t))$ remove the exponential function $e^{\alpha(s-t)}$. By adding $\alpha V_3(x(t))$, the inequality (3.2.14) satisfies the condition of Lemma 2.1.5. The novel techniques has had the effect of producing less conservative results in the reachable set of (3.2.1).

Remark 10 In [46], a simple approximation to minimize the volume ϵ of (3.2.6) was suggested as $\delta I \leq P$ and this is equivalent as follows:

$$\begin{aligned} & \text{minimize } \bar{\delta} = \frac{1}{\delta}, \\ & \text{subject to } \begin{bmatrix} \bar{\delta} I & I \\ I & P \end{bmatrix}, \text{ and LMIs (3.2.7) and (3.2.8).} \end{aligned} \quad (3.2.34)$$

Likewise [47], the optimization toolbox is used to solve the optimization value of α by fminsearch.m. The obtained results can be used to confirm the ellipsoid in the following numerical examples.

The following Theorem 3.2.2 shows an improved result than the results of Theorem 3.2.1.

Theorem 3.2.2 *For given scalars h_M and μ , system (3.2.1) is asymptotically stable for $0 \leq h(t) \leq h_M$ and $\dot{h}(t) \leq \mu$, if there exist scalar $\alpha > 0$, matrices $U_m \in \mathbb{S}_+^n$, $P, R, Q_1 \in \mathbb{S}_+^{3n}$, $Q_2 \in \mathbb{S}_+^{2n}$, $\Lambda_i, \Delta_i \in \mathbb{D}_+^n$, $P_i \in \mathbb{S}^n$, $S_i, N_i \in \mathbb{R}^{19n-l \times 3n}$, $H_j, F_j \in \mathbb{D}_+^n$ ($j = 1, 2, \dots, 6$) satisfying the following LMIs:*

$$\begin{bmatrix} \left(\Gamma_{m[h]}^\perp\right)^T \left(\Xi_{[h_M]}^m - \beta e_{19} e_{19}^T\right) \left(\Gamma_{m[h]}^\perp\right) \left(\Gamma_{m[h]}^\perp\right)^T (h_M S_1) \left(\Gamma_{m[h]}^\perp\right)^T (h_M N_1) \\ \star & -h_M e^{\alpha h_M} R_{aug1} & 0_{3n} \\ \star & \star & -3h_M e^{\alpha h_M} R_{aug1} \end{bmatrix} < 0, \quad (3.2.35)$$

$$\begin{bmatrix} \left(\Gamma_{m[0]}^\perp\right)^T \left(\Xi_{[0]}^m - \beta e_{19} e_{19}^T\right) \left(\Gamma_{m[0]}^\perp\right) \left(\Gamma_{m[0]}^\perp\right)^T (h_M S_2) \left(\Gamma_{m[0]}^\perp\right)^T (h_M N_2) \\ \star & -h_M e^{\alpha h_M} R_{aug2} & 0_{3n} \\ \star & \star & -3h_M e^{\alpha h_M} R_{aug2} \end{bmatrix} < 0, \quad (3.2.36)$$

$$R_{aug1} \geq 0, \quad (3.2.37)$$

$$R_{aug2} \geq 0, \quad (3.2.38)$$

where

$$\Gamma_{m[h(t)]} = \begin{bmatrix} \Gamma \\ h(t)e_6^T - e_{17}^T \\ (h_M - h(t))e_7^T - e_{18}^T \end{bmatrix}. \quad (3.2.39)$$

The same notations are used, like Theorem 3.2.1.

Proof Let us see the stability condition of the LKFs for a system (3.2.1). It can be written by

$$\begin{aligned} \mathfrak{L}\{V(t)\} &\leq \zeta^T(t) \left[\Xi_{[h(t)]} + h(t) \left\{ S_1(R_{aug1})^{-1} S_1^T + \frac{1}{3} U_1(R_{aug1})^{-1} U_1^T \right\} \right. \\ &\quad \left. + (h_M - h(t)) \left\{ S_2(R_{aug2})^{-1} S_2^T + \frac{1}{3} U_2(R_{aug2})^{-1} U_2^T \right\} \right. \\ &\quad \left. - \beta e_{19} e_{19}^T \right] \zeta(t) < 0, \\ &\text{subject to } 0 = \Gamma_{[h(t)]} \zeta(t). \end{aligned} \quad (3.2.40)$$

By (i) and (ii) of Lemma 2.1.3, the above condition is equivalent to

$$\begin{aligned} &\Xi_{[h(t)]} + h(t) \left\{ S_1(Q_1^{aug})^{-1} S_1^T + \frac{1}{3} U_1(Q_1^{aug})^{-1} U_1^T \right\} \\ &+ (h_M - h(t)) \left\{ S_2(Q_2^{aug})^{-1} S_2^T + \frac{1}{3} U_2(Q_2^{aug})^{-1} U_2^T \right\} \\ &+ Sym \{ M \Gamma_{[h(t)]} \} - \beta e_{19} e_{19}^T < 0, \end{aligned} \quad (3.2.41)$$

for any matrix M , where $\Xi_{[h(t)]}$ is the same notation in Theorem 3.2.1.

The inequality (3.2.41) is satisfied, if the following two inequalities hold

$$\begin{aligned} &\Xi_{[h_M]} + h_M \left\{ S_1(Q_1^{aug})^{-1} S_1^T + \frac{1}{3} U_1(Q_1^{aug})^{-1} U_1^T \right\} \\ &+ Sym \{ M \Gamma_{[h_M]} \} - \beta e_{19} e_{19}^T < 0, \end{aligned} \quad (3.2.42)$$

$$\begin{aligned} &\Xi_{[0]} + h_M \left\{ S_2(Q_2^{aug})^{-1} S_2^T + \frac{1}{3} U_2(Q_2^{aug})^{-1} U_2^T \right\} \\ &+ Sym \{ M \Gamma_{[0]} \} - \beta e_{19} e_{19}^T < 0, \end{aligned} \quad (3.2.43)$$

since the above inequality relies on $h(t)$ with the condition of $0 \leq h(t) \leq h_M$. By utilizing (ii) and (iii) of Lemma 2.1.3, it can be confirmed that if inequalities (3.2.35) and (3.2.36) hold, then the inequality (3.2.41) satisfied for the system (3.2.1). Since

the other proof is the same as Theorem 3.2.1, it can be omitted. ■

Remark 11 Zero equalities have some drawbacks, which increase the calculating costs due to the free variables $\Psi_i \in \mathbb{R}^{n \times (19n-l)}$ ($i = 1, 2$). For solving these problems, the AZEA was proposed in [35]. All the zero equalities such that

$$0 = (-A(r_t)e_1^T - e_4^T + W_0(r_t)e_8^T + W_2(r_t)e_9^T + B(r_t)e_{19}^T) \zeta(t),$$

$$0 = (h(t)e_6 - e_{17})\zeta(t),$$

$$0 = ((h_M - h(t))e_7 - e_{18})\zeta(t),$$

in Theorem 3.2.1 are incorporated in $0 = \Gamma_{[h(t)]}\zeta(t)$. By utilizing the AZEA, Lemma 2.1.4 was applied to (3.2.40). Consequently, it is possible not only to eliminate the free variables $\Psi_i \in \mathbb{R}^{n \times (19n-l)}$ ($i = 1, 2$), but also to guarantee less conservatism in stability criteria with improved maximum upper bounds of time-varying delays.

3.2.3 Numerical Examples

Two examples verify that the proposed techniques can be effective and estimate bounds of smaller reachable set in this paper.

Example 3.2.1 Let us consider the system (3.2.1) with $r_t \in \{1, 2\}$:

$$\begin{aligned} A(1) &= \begin{bmatrix} 2 & 0 \\ 0 & 3.5 \end{bmatrix}, \quad A(2) = \begin{bmatrix} 2.2 & 0 \\ 0 & 1.8 \end{bmatrix}, \quad B(1) = B(2) = \begin{bmatrix} 0.0403 \\ 0.6771 \end{bmatrix}, \\ W_0(1) &= \begin{bmatrix} -1 & 0.5 \\ 0.5 & -1 \end{bmatrix}, \quad W_0(2) = \begin{bmatrix} -1 & 1 \\ 0.5 & -1 \end{bmatrix}, \quad W_1 = I_2, \\ W_2(1) &= \begin{bmatrix} -0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}, \quad W_2(2) = \begin{bmatrix} -0.5 & 0.6 \\ 0.7 & 0.8 \end{bmatrix}, \end{aligned}$$

$$K_p = \text{diag}\{0.4, 0.8\}, \quad K_m = 0_2. \quad (3.2.44)$$

The transition probabilities matrix is given by $\Pi = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix} = \begin{bmatrix} -0.2 & 0.2 \\ 0.8 & -0.8 \end{bmatrix}$. The external input of Markovian jump NNTVDs (3.2.1) is assumed to satisfy (3.2.3) with $\omega_M = 1$.

In Table 3.2.1 and Table 3.2.2, allowable smaller bounds $\bar{\delta}$ are shown for different values h_D . The minimum bounds of the reachable set obtained by optimizing $\bar{\delta}$ for α are listed in Table 3.2.1 and Table 3.2.2. They includes the results of a recent research such as [44]. By comparing the results, it is explicitly verified that the proposed methods are enhanced. μ is smaller and smaller, the allowable minimum bounds of $\bar{\delta}$ show impactful performances in the reachable set of Markovian jump NNTVDs. To check the effectiveness of the results in (3.2.1) with $h = 3$ and $\mu = 2$, the random signals satisfying time-varying delays $h(t) = 1.5\sin\left(\frac{2}{15}t\right) + 1.5$, the activation function $f(x(t)) = \begin{bmatrix} 0.4\tanh(x_1(t)) & 0.8\tanh(x_2(t)) \end{bmatrix}^T$, and choosing disturbances $\omega(t)$ as the random signal satisfying $|\omega(t)| \leq 1$ are illustrated in Fig. 3.2.1. When the optimal $\bar{\delta} = 0.7018, \alpha = 0.2998$ of Theorem 2 are obtained by solving optimization problem, the corresponding feasible matrices U_m are derived as $U_1 = \begin{bmatrix} 1.5085 & -0.0028 \\ -0.0028 & 1.9716 \end{bmatrix}, U_2 = \begin{bmatrix} 1.4249 & 0 \\ 0 & 1.4249 \end{bmatrix}$. All the reachable states and the bounds ellipsoids of (3.2.1) starting from origin are shown in Fig. 3.2.2. It can be clearly seen that all the reachable states of (3.2.1) are surrounded by the reachable set obtained by this paper, which illustrate the effectiveness of our derived results.

Table 3.2.1: Allowable smaller bounds $\bar{\delta}$ for various μ and $h_M = 2$ (Example 3.2.1).

μ	0.2	0.4	0.6	0.8
W.J. Lin <i>et al.</i> [44]	0.6821 ($\alpha = 0.44$)	0.8578 ($\alpha = 0.39$)	1.2302 ($\alpha = 0.31$)	2.2204 ($\alpha = 0.22$)
Ours (Theorem 1)	0.5158 ($\alpha = 0.4124$)	0.5965 ($\alpha = 0.3817$)	0.7221 ($\alpha = 0.3526$)	0.8412 ($\alpha = 0.3252$)
Ours (Theorem 2)	0.4698 ($\alpha = 0.4462$)	0.5031 ($\alpha = 0.4550$)	0.5301 ($\alpha = 0.4624$)	0.5469 ($\alpha = 0.4594$)

Table 3.2.2: Allowable smaller bounds $\bar{\delta}$ for various μ and $h_M = 3$ (Example 3.2.1).

μ	0.2	0.4	0.6	0.8
W.J. Lin <i>et al.</i> [44]	0.9443 ($\alpha = 0.30$)	1.2456 ($\alpha = 0.26$)	1.9351 ($\alpha = 0.21$)	5.0915 ($\alpha = 0.12$)
Ours (Theorem 1)	0.7480 ($\alpha = 0.2807$)	0.9034 ($\alpha = 0.2516$)	1.2175 ($\alpha = 0.2156$)	1.9624 ($\alpha = 0.1698$)
Ours (Theorem 2)	0.7018 ($\alpha = 0.2998$)	0.7939 ($\alpha = 0.2875$)	0.9274 ($\alpha = 0.2702$)	1.1454 ($\alpha = 0.2437$)

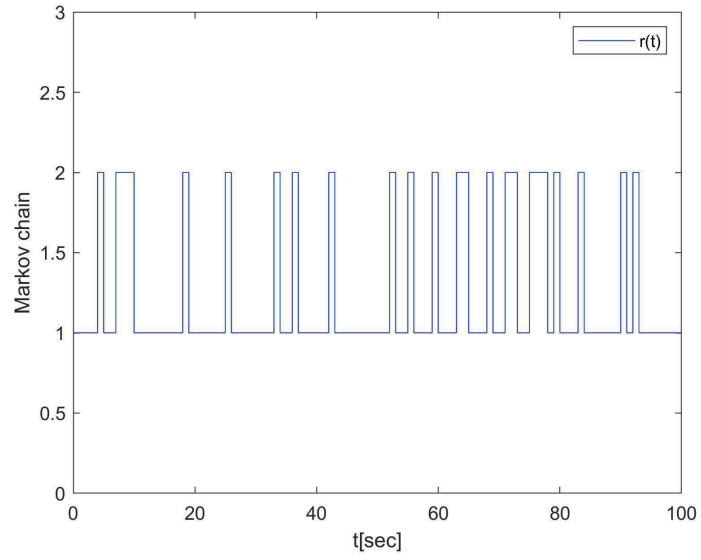


Figure 3.2.1: Random jumping mode $r(t)$ of (3.2.1) (Example 3.2.1).

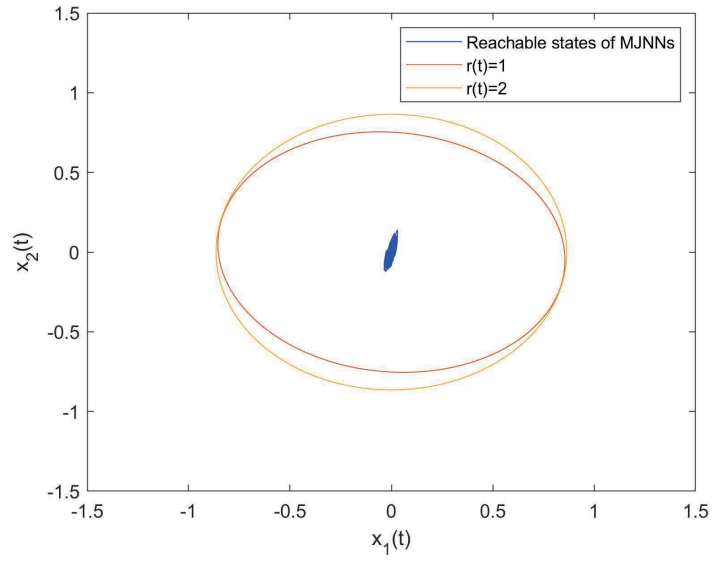


Figure 3.2.2: Reachable states of (3.2.1) (Example 3.2.1).

Example 3.2.2 Let us consider the system (3.2.1) with $r_t \in \{1, 2\}$:

$$\begin{aligned}
A(1) &= \begin{bmatrix} 2.5 & 0 \\ 0 & 0.4 \end{bmatrix}, \quad A(2) = \begin{bmatrix} 2 & 0 \\ 0 & 1.5 \end{bmatrix}, \quad B(1) = \begin{bmatrix} 0.0513 \\ 0.5372 \end{bmatrix}, \quad B(2) = \begin{bmatrix} 0.0303 \\ 0.4996 \end{bmatrix}, \\
W_0(1) &= \begin{bmatrix} -1.5 & 0.1 \\ 0.1 & -1.5 \end{bmatrix}, \quad W_0(2) = \begin{bmatrix} -1.5 & 1 \\ -0.5 & -1.5 \end{bmatrix}, \quad W_1 = I_2, \\
W_2(1) &= \begin{bmatrix} -1 & 1.5 \\ 1.5 & 0.5 \end{bmatrix}, \quad W_2(2) = \begin{bmatrix} -0.5 & 0.5 \\ 0.6 & 0.7 \end{bmatrix}, \\
K_p &= \text{diag}\{0.5, 0.5\}, \quad K_m = 0_2.
\end{aligned} \tag{3.2.45}$$

The transition probabilities matrix is given by $\Pi = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix} = \begin{bmatrix} -0.9 & 0.9 \\ 0.3 & -0.3 \end{bmatrix}$. The external input of Markovian jump NNTVDs (3.2.1) is assumed to satisfy (3.2.3) with $\omega_M = 1$.

The obtained bounds $\bar{\delta}$ are listed by the conditions of different values h_D and $\alpha = 0.1894$ in Table 3.2.3. They includes the results of a recent research such as [45]. By comparing the results, it is explicitly verified that the proposed methods are enhanced. According to various μ , the allowable minimum bounds of $\bar{\delta}$ show less conservative in the reachable set of (3.2.1).

Table 3.2.3: Allowable smaller bounds $\bar{\delta}$ for various μ ($h_M = 3$, $\alpha = 0.1894$) (Example 3.2.2).

μ	0.2	0.4	0.6	0.8
W. J. Lin <i>et al.</i> [44]	0.5192	0.5386	0.5836	0.7929
X. Jiang <i>et al.</i> [45]	0.5173	0.5364	0.5807	0.7789
Ours (Theorem 1)	0.4954	0.5044	0.5228	0.5839
Ours (Theorem 2)	0.4917	0.4991	0.5080	0.5372

3.2.4 Conclusion

In this paper, enhanced delay-dependent the estimation of the reachable set for Markovian jump NNTVDs are proposed. By constructing the improved augmented LKFs expanded the previous researches and utilizing GFWMI, the optimization of the reachable set parameter for delayed Markovian jump neural networks was illustrated in Theorem 3.2.1. Based on Theorem 3.2.1, the AEZA in [35] is applied to a more relaxed criterion, eliminating free-variables caused zero equality and expanding the maximum delay bounds. Two numerical examples are used to confirm an improved results and the minimum of the ellipsoid bounds in a reachable set. In Figure 3.2.1 and Figure 3.2.2, it shows the random jumping mode and the reachable states of Markovian jump NNTVDs, respectively.

IV. Conclusions

In this thesis, enhanced delay-dependent stability criteria for Generalized NNTVDs system and delay-dependent the estimation of the reachable set for Markovian jump NNTVDs are investigated. Firstly, to obtain less conservative results, a newly augmented Lyapunov-Krasovskii's functional such as including the activation functions term is used to improve the feasible region of stability analysis. Secondly, the estimation of the reachable set is proposed by optimizing the parameter based on the principles of stability analysis. Finally, numerical examples are used to verify the validity and superiority than previous literature. The predicting the maximum delay bounds, the results are checked by the graph as shown in Figure 3.1.1. In Figure 3.2.1 and Figure 3.2.2, it shows the random jumping mode and the reachable states of Markovian jump NNTVDs, respectively. The proposed stability analysis and estimation of the reachable set for the neural networks systems are represented by the Lyapunov method. All simulations use MATLAB LMI Toolbox and Simulink.

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